Oklahoma State University

Number Theory Seminar

Title

Noncommutative Diophantine Approximation

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Date: Apr 5, 2016
Time: 3:30 PM
Room: MSCS 509

Abstract: Let G be a compact topological group that acts continuously and faithfully on a compact metric space $\{X, m\}$. If $\mathcal{A} \subseteq G$ is a finite subset containing at least two points, we describe two results that establish the existence of a nonidentity element in the difference set

$$\left\{ab^{-1} : a \in \mathcal{A},\ b \in \mathcal{A},\ \text{and } a \neq b\right\}$$

that moves the points of X minimally. We consider explicit results of this sort in the case of the unitary group $G = U(N)$ of $N \times N$ unitary matrices acting on the complex surface

$$X = \{x \in \mathbb{C}^N : |x|_2 = 1\}.$$

We define a function $\varphi : U(N) \to [0, \infty)$ by

$$\varphi(A) = \sup\left\{|Ax - x|_2 : x \in X\right\},$$

so that $\varphi(A)$ measures the maximum distance that A moves a point x in X.
Theorem 1. Let $\mathcal{A} \subseteq U(N)$ be a finite subset of cardinality $|\mathcal{A}| \geq 2$. If

$$\delta(\mathcal{A}) = \min\left\{\varphi(AB^{-1}) : A \in \mathcal{A},\ B \in \mathcal{A},\ \text{and } A \neq B\right\},$$

then we have

$$\delta(\mathcal{A}) \leq 2\pi|\mathcal{A}|^{-1/N^2}.$$

As an application of Theorem 1, we obtain the following noncommutative approximation theorem.
Theorem 2. Let A and B be matrices in the unitary group $U(N)$, and let J and K be positive integers. If

$$\delta_{J,K}(A, B) = \min\left\{\varphi(A^j B^k) : |j| \leq J,\ |k| \leq K, \text{ and } (j, k) \neq (0, 0)\right\},$$

then

$$\delta_{J,K}(A, B) \leq 2\pi(J + 1)^{-1/N^2}(K + 1)^{-1/N^2}.$$

This is joint work with Clay Petsche.