Abstract: The purpose of this talk is two-fold. First, it is meant as an expository talk, accessible to graduate students, for the purpose of gently introducing Schubert varieties, the Bott-Samelson resolution, and more generally, Bott-Samelson varieties. Second, it is meant as a prelude to a second talk (to be given in the Lie Groups Seminar on Nov. 8) where I will discuss recent joint work with Laura Escobar and Alex Yong, in which we generalize some of the results I will describe in this talk.

Schubert varieties in the complex flag variety are singular in general. By a general theorem of Hironaka, any singular Schubert variety will admit a resolution of singularities. However, Hironaka’s construction is usually horrible to work with, so for the purpose of computation, it is often useful to have an explicit resolution on hand. The Bott-Samelson resolution is such an explicit resolution. Roughly speaking, there is a Bott-Samelson resolution for each reduced word of the Weyl group element indexing the Schubert variety in question, with the total space of the resolution being constructed by starting with a point and then iteratively constructing $P^1$-bundles over it. If we mimic this construction while dropping the requirement that the word be reduced, we get a more general object, called a Bott-Samelson variety. This variety is still smooth (still being an iterated $P^1$-bundle over a point), and it still maps to the Schubert variety, but the map is no longer birational if the word is non-reduced.

While it is explicit, the Bott-Samelson variety can also be a bit abstract and difficult to picture. I will discuss a different realization of it as an iterated fiber product, due to P. Magyar, which has the benefit of being a bit more concrete. Indeed, in type A we can even parametrize its points by simple diagrams; this sort of thing is very attractive to those who like combinatorics. Besides the added concreteness, the alternative formulation also makes clear that the Bott-Samelson variety is a symplectic manifold with a Hamiltonian torus action, and as such, has an associated moment polytope. I will discuss some known results about this polytope, as well as the moment polytope of a certain subvariety of the Bott-Samelson variety, called the brick variety.