MASTER'S COMPREHENSIVE EXAM—ADVANCED CALCULUS
AUGUST, 1989

1. Let A, B be bounded nonempty sets of real numbers. Define $C = \{a + b \mid a \in A, b \in B\}$. Show that $\sup C = \sup A + \sup B$.

2. Find all values of x for which $\sum_{n=1}^{\infty} a_n x^n$ converges, where $a_n = \begin{cases} (1/2)^{(n+1)/2} & \text{if } n \text{ is odd} \\ (1/3)^{n/2} & \text{if } n \text{ is even.} \end{cases}$

3. Let $f : \mathbf{R} \to \mathbf{R}$ be defined by

$$f(x) = \begin{cases} x + 2x^2 \sin\left(\dfrac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

   (a) Find $f'(0)$.
   (b) Is f continuously differentiable on $\mathbf{R}$?

4. Let $f : \mathbf{R}^2 \to \mathbf{R}$ be continuously differentiable on $\mathbf{R}^2$ and suppose that f satisfies $(D_1 f)(-2, 1) = 2$, $(D_2 f)(-2, 1) = 3$. Define $g : \mathbf{R}^3 \to \mathbf{R}$ by

$$g(x, y, z) = f(xy + xz + yz, e^{xyz}).$$

   find $(D_i g)(2, -1, 0)$ for $i = 1, 2, 3$.

5. Determine all values of h for which

$$\sum_{n=1}^{\infty} \frac{x}{n(4 + x^2 n)}$$

   converges uniformly on $[-h, h]$. Does this series converge uniformly on $(-\infty, \infty)$?

6. Let f be a function of two variables for which $f(1, 0) = 1$. What additional conditions on f will allow the equation

$$f(xy, xz) + f(e^z, yz) = 2$$

   to be solved for z in a neighborhood of $(1, 1, 0)$?

7. Evaluate $\int_D \exp\left(\dfrac{x - y}{x + y}\right) dx\, dy$, where D is the triangle bounded by the lines $x = 0$, $y = 0$, $x + y = 1$. (Hint: Change of variable $u = x + y$, $v = x - y$.)

8. Evaluate $\int_C (y^2 + \sin(x^2))dx + (\cos(y^2) - x)dy$, where C is the boundary of the unit square $0 \leq x \leq 1$, $0 \leq y \leq 1$. (Hint: Green's Theorem)