Ph.D. Comprehensive Exam

Real Analysis – August 1990

PART I: Do all four problems.

1. In the interval [0,1] construct: (a) an uncountable set of Lebesgue measure 0. (b) a set of positive Lebesgue measure having no interior.

2. (a) State the Monotone Convergence Theorem, Fatou's Lemma, and Lebesgue's Dominated Convergence Theorem.

(b) Prove Lebesgue's Dominated Convergence Theorem. (You may use the other two if you need to.)

3. (a) Suppose $(X, \mathcal{M})$ is a measure space and μ, ν are measures on $(X, \mathcal{M})$. State the Radon-Nikodym Theorem for μ and ν.

(b) Give an example of a measure on $\mathbb{R}$ which is singular with respect to the Lebesgue measure on $\mathbb{R}$.

(c) Give an example of a non-trivial measure on $\mathbb{R}$ which is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$. What is its Radon-Nikodym derivative with respect to Lebesgue measure?

4.(a) State carefully the Riesz Representation Theorems describing the bounded linear functionals on $L^p(\mu)$, $1 \leq p < \infty$ and $C(K)$. (Here K is a compact Hausdorff space.)

(b) If $I = [0,1]$, give an example of a linear functional on $C(I)$ which is not represented by an element of $L^1(I)$. Explain.

PART II: Do two of the following four problems.

1. Let f be a real valued Lebesgue measurable function on $[0,1]$. Prove that for any $\epsilon > 0$ there exists a real valued continuous function g defined on $[0,1]$ such that $m(\{x \in [0,1] | f(x) \neq g(x)\}) < \epsilon$, where m is Lebesgue measure on $[0,1]$.

2. If μ is a Borel measure on $\mathbb{R}$ such that $\mu([0,1]) = 1$ and for each $x \in \mathbb{R}$ and each Borel set E

$$\mu(E) = \mu(x + E)$$

show that μ is Lebesgue measure restricted to $\mathcal{B}_{\mathbb{R}}$, the sigma algebra of Borel sets on $\mathbb{R}$..

3. Suppose $f : \mathbb{R} \longrightarrow \mathbb{R}$ is measurable, suppose $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$, and suppose that for any $g \in L^q(\mathbb{R})$, $f \cdot g \in L^1(\mathbb{R})$. Prove that f belongs to $L^p(\mathbb{R})$.

4. Suppose $f \in L^1(\mathbb{R})$ and $g \in L^p(\mathbb{R})$ for $1 \leq p < \infty$. Define their convolution by

$$f * g(x) = \int f(x - y) g(y)\, dy.$$

Prove that $f * g \in L^p(\mathbb{R})$ and $\|f * g\|_p \leq \|f\|_1 \|g\|_p$.