Introduction to Modern Algebra

1. Let G be a group with the unit element 1. Suppose N_1, N_2 are normal subgroups of G and $N_1 \cap N_2 = \{1\}$. Prove that for any $n_1 \in N_1, n_2 \in N_2$, $n_1 n_2 = n_2 n_1$.

2. Let H_1, H_2 be two Sylow-5 subgroups of S_7 (the permutation group of a set of 7 elements). Prove that H_1, H_2 are conjugate i.e. there exists $r \in S_7$ such that $H_1 = rH_2r^{-1}$.

3. Let R be a commutative ring with the unit element 1, $I \subset R$ and ideal, and $r \in R$. Define

$$I : r = \{x \in R \mid xr \in I\}.$$

(1) Prove that $I : r$ is an ideal of R.

(2) Prove that the submodule $R(r + I)/I \subset R/I$ is isomorphic to $R/(I : r)$ as a module over R.

4. Consider $\mathbb{C}^2$ as a vector space over $\mathbb{R}$. Define $f : \mathbb{C}^2 \to \mathbb{C}^2$ by

$$f : (a, b) \to (a + \bar{b}, a - \bar{b}).$$

Find the dimensions of the image and the kernel of this linear transformation over $\mathbb{R}$.

5. Let K be the splitting field of the polynomial $x^{63} - 1$ over $\mathbb{Q}$. Find the number of all the fields F's such that $F \subset K$ and $[F : \mathbb{Q}] = 2$.