Algebra Examination (Ph.D.)

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1. Let p and q be prime numbers with $q < p$.

   (a) Prove that any finite group of order pq is solvable. (Use of the Sylow theorems is permitted.)

   (b) Prove that the symmetric group S_p of degree p is generated by any 2-cycle and any p-cycle.

   (c) Give a counterexample to part (b) if p is not prime.

2. Let R be a ring, and let N_1 and N_2 be submodules of the left R-module M such that $N_1 + N_2 = M$. Prove that $M/(N_1 \cap N_2)$ is isomorphic to $(M/N_1) \oplus (M/N_2)$ as left R-modules.

3. Let A be a commutative ring with identity, and let $\mathfrak{a}$ be a proper ideal of A.

   (a) Prove that $\mathfrak{a}$ is contained in a maximal ideal $\mathfrak{m}$ of A.

   (b) Let S be a multiplicative subset of A such that $S \cap \mathfrak{a} = \emptyset$. Prove that there is an ideal $\mathfrak{p}$ which is maximal subject to the conditions that $\mathfrak{a} \subset \mathfrak{p}$ and $\mathfrak{p} \cap S = \emptyset$. Prove that $\mathfrak{p}$ is a prime ideal of A.

   (c) Let $\mathfrak{a} = \bigcap_{\mathfrak{p}} \mathfrak{p}$ be the intersection of all prime ideals of A. Prove that $\mathfrak{a}$ consists precisely of all elements x of A such that $x^n = 0$ for some integer $n > 0$.

   (d) Suppose now that the ring A is Noetherian as well. Prove that the ideal $\mathfrak{a}$ considered in part (c) is nilpotent.

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4. Let $f(x) = x^4 - 7$.

   (a) Determine the degree of the splitting field K of $f(x)$ over $\mathbb{Q}$.

   (b) Determine all the subextensions F such that $\mathbb{Q} \subset F \subset K$, and arrange these in a graph ordered by inclusion.

5. (a) Define what it means for an irreducible polynomial $f(x) \in \mathbb{Z}[x]$ to be solvable by radicals over $\mathbb{Q}$.

   (b) Prove that $x^5 - 9x + 3$ is irreducible.

   (c) Prove that $x^5 - 9x + 3$ is not solvable by radicals over $\mathbb{Q}$.