Complex Variables I & II
August 1992

Show all work and justify your answers.

1. Let z_0 be a point in the extended complex plane $\mathbb{C}^*$.
   a. Define the three types of isolated singularities of a function f analytic in $\{z : r > |z - z_0| > 0\}$ for some $r > 0$.
   b. Suppose that $r(z)$ is a rational function, i.e., $r(z) = \dfrac{p(z)}{q(z)}$ where $p(z)$ and $q(z)$ are polynomials. Is it possible for $r(z)$ to have an essential singularity at $z_0 = \infty$. Why or why not?
   c. Assume that ∞ is not an essential singularity of the rational function $r(z)$. How can one determine the type of singularity at ∞?
   d. Sketch the proof that if $f(z)$ is analytic in a disk $\{z : |z - z_0| < r\}$ for some $r > 0$, then the Taylor's series for f about z_0 converges at least in this same disk.
   e. Suppose that the rational function $r(z)$ is defined at z_0. Explain how one could find the exact radius of convergence of the power series for $r(z)$ with expansion point z_0 without determining the coefficients of the power series.

2. Suppose that $f(z)$ is non-constant and analytic in some region R and that $z_0 \in R$.
   a. Define the winding number of a closed rectifiable curve γ in R about z_0.
   b. Show that $\displaystyle\lim_{r\to 0^+} (2\pi i)^{-1} \int_{\gamma_r} \frac{f'(z)}{f(z)} dz$ exists where γ_r is circle of radius r with center at z_0 traversed once in the counterclockwise direction.
   c. For $w \in R$ define $g(w) = \displaystyle\lim_{r\to 0^+} (2\pi i)^{-1} \int_{\gamma_r} \frac{f'(z)}{f(z)} dz$. What condition on f is necessary for g to be analytic in R?

3. Suppose that f is analytic in the open unit disk D and that $|f(z)| \le |z|$ for all $z \in D$. Prove that if $f(0) = f'(0) = 0$ and $f(1/2) = i/4$ then $f(z) = iz^2$.

4. Suppose that G is a region in $\mathbb{C}$ and that $\mathcal{F}$ is a family of complex valued functions on G.
   a. Define "$\mathcal{F}$ is a normal family".
   b. Prove that if $\mathcal{F}$ is a normal family of analytic functions on G, then $\mathcal{F}' = \{f' : f \in \mathcal{F}\}$ is also a normal family.

5. Let (z_n) be a sequence in $\mathbb{C}$.
   a. Define $\prod z_n$ converges.
   b. Suppose that $z_n = 2^n$. Show that there is an entire function g with a zero of order one at z_n and no other zeroes. Explicitly define such a function.