MASTER'S COMPREHENSIVE EXAM—ADVANCED CALCULUS
AUGUST, 1993

1. Let A be a bounded nonempty set of real numbers. Define $-A = \{-a \mid a \in A\}$. Show that
$$\inf(-A) = -\sup A.$$

2. Find all values of x for which $\sum_{n=1}^{\infty} a_n x^n$ converges, where $a_n = \begin{cases} \sin n & \text{if } n \text{ is odd} \\ 4^{n/2} & \text{if } n \text{ is even.} \end{cases}$

3. Fix a real number r and let $f : \mathbf{R} \to \mathbf{R}$ be defined by
$$f(x) = \begin{cases} \dfrac{|x|^r}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$
(a) For what values of r is f continuous on $\mathbf{R}$?
(b) For what values of r is f differentiable on $\mathbf{R}$ and what is $f'(0)$?
(c) For what values of r is f continuously differentiable on $\mathbf{R}$?

4. Let $f : \mathbf{R}^2 \to \mathbf{R}$ be of class C^2 on $\mathbf{R}^2$ and define $g : \mathbf{R}^2 \to \mathbf{R}$ by the formula
$$g(x, y) = f(x^2 - y^2, y^2 - x^2).$$
Express $D_1 g$ and $D_{11} g$ in terms of the $D_i f$ and the $D_{ij} f$, $i, j = 1, 2$.

5. Consider the series $\sum_{n=1}^{\infty} (xe^{-x})^n$.
(a) For which $R > 0$ does the series converge uniformly on $[0, R]$?
(b) Does the series converge uniformly on $[0, \infty)$?
(c) For which $R < 0$ does the series converge uniformly on $[R, 0]$?

6. Let f and g be functions on $\mathbf{R}^2$ such that $f(1, 1) = g(1, 1) = 1$. Find conditions on f and g that permit you to solve the equation
$$f(xy, yz)g(xy, yz) = 1$$
for y as a differentiable function of x and z near the point $x = y = z = 1$.

7. Evaluate $\displaystyle\int_D \frac{x + y}{xy}\, dx\, dy$, where D is the region in the first quadrant bounded by the curves $xy = 1$, $xy = 3$, $y - x = -1$, $y - x = 1$. (Hint: Make a change of variable to simplify the description of D.)

8. Evaluate $\int_C (y^2 + \sin(x^2))dx + (\cos(y^2) - x)dy$, where C is the boundary of the unit square $0 \leq x \leq 1$, $0 \leq y \leq 1$. (Hint: Green's Theorem)