PhD COMPREHENSIVE EXAM—COMPLEX ANALYSIS—August 1993

Notation: $B = \{z \in \mathbf{C}\colon |z| < 1\}$, the open unit disc

1. Show that the real and imaginary parts of an analytic function are harmonic.
2. Show that the polynomial z^5+5z^3+z-2 has exactly three zeros (counting multiplicities) in B.
3. Let f be an entire function, and suppose there is a finite constant M so that
$$\int_0^{2\pi} |f(re^{i\theta})|\, d\theta \le M$$
for all $r > 0$. Show that f is constant by finding $f^{(n)}(0)$ for all $n \ge 1$.
4. Let $\mathcal{F}$ denote the family of functions f analytic on B so that $f(0) = 0$ and
$$|f'(z)| \le \frac{1}{1-|z|}$$
for all $z \in B$. Show that $\mathcal{F}$ is a normal family.
5. a) Define, for $|z| > 1$,
$$f(z) = \frac{2}{z} + \sum_{n=2}^{\infty} \frac{1}{z^n} \quad .$$
Find an analytic continuation of f to the set $D = \{z\colon z \ne 0 \text{ and } z \ne 1\}$; that is, find a function g analytic on D so that $g(z) = f(z)$ if $|z| > 1$.

   b) With notation as in a), how many such analytic continuations of f to D are there? Why?

   c) If g is as in a), what is the residue of g at 0?
6. a) Does there exist a one-to-one analytic function f mapping the upper half plane $H = \{x + iy\colon -\infty < x < \infty \text{ and } y > 0\}$ onto B? If so, can f be chosen to have an analytic continuation to a neighborhood of the closure of H in $\mathbf{C}$? Explain your answers.

   b) Repeat a) with B replaced by the square $S = \{x + iy\colon 0 < x < 1 \text{ and } 0 < y < 1\}$.