MASTER'S COMPREHENSIVE EXAM—ADVANCED CALCULUS
AUGUST, 1994

1. Let A and B be nonempty sets of real numbers such that $a < b$ for all $a \in A$ and all $b \in B$. Show that $\sup(A) \leq \inf(B)$.

2. Find all values of x for which the series $\displaystyle\sum_{n=0}^{\infty} \left(x \cos \frac{n\pi}{3}\right)^n$ converges.

3. Let a be a positive integer and define $f : \mathbf{R} \to \mathbf{R}$ by

$$f(x) = \begin{cases} x^3 \sin \dfrac{1}{x^a} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

   (a) For what values of a is f differentiable at 0 and what is $f'(0)$?
   (b) For what values of a is f' continuous at 0?

4. Let $f : \mathbf{R}^2 \to \mathbf{R}$ be continuously differentiable on $\mathbf{R}^2$ and suppose that f satisfies $(D_1 f)(1, -2) = 2$, $(D_2 f)(1, -2) = 3$. Define $g : \mathbf{R}^3 \to \mathbf{R}$ by the formula

$$g(x, y, z) = f(xyz, -2\cos(xy + xz + yz + 1)).$$

   Find $(D_i g)(-1, 1, -1)$ for $i = 1, 2, 3$.

5. Find all values of h for which the series $\displaystyle\sum_{n=1}^{\infty} \frac{x}{n(4 + x^2 n)}$ converges uniformly on $[-h, h]$. Does it converge uniformly on $(-\infty, \infty)$?

6. Given $f : \mathbf{R}^2 \to \mathbf{R}$ satisfying $f(-2, 1) = 1$, what additional conditions on f will allow you to solve the equation

$$f(xy + xz + yz, e^{xyz}) = 1$$

   for z as a function of x and y in a neighborhood of the point $(2, -1, 0)$?

7. Let D be the rectangular region in the plane

$$D = \{(u, v) \in \mathbf{R}^2 \mid 0 \leq u \leq 3 \text{ and } 1 \leq v \leq 3\}$$

   and let $T : \mathbf{R}^2 \to \mathbf{R}^2$ be the transformation

$$T(u, v) = (u, v(1 + 2u)).$$

   (a) Sketch $T(D)$ and find the equations of the curves that form its boundary.
   (b) Evaluate $\displaystyle\int_{T(D)} \frac{dx\,dy}{1 + 2x}$.

8. Evaluate $\int_C (y + e^{x^2})dx + (\ln(1 + y^2) - x^2)dy$, where C is the boundary of the unit square $0 \leq x \leq 1$, $0 \leq y \leq 1$ oriented counterclockwise. (Hint: Green's Theorem)