Master's Comprehensive Exam
Complex Variables
August 1994

1. (10 pts.) Evaluate the following:

   (a) $\sin\left(\frac{\pi}{2} - 3i\right)$

   (b) $\mathrm{Log}\left(\frac{-1 - i\sqrt{3}}{2}\right)$, where Log represents the Principal branch of the logarithm.

2. (16 pts.) Let $f(z) = z\mathrm{Im}(z)$.

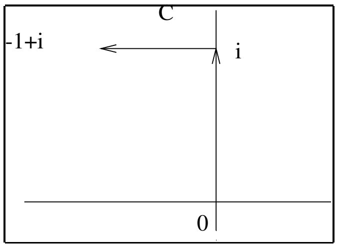
   (a) What is the domain of $f(z)$?

   (b) Where is $f(z)$ continuous?

   (c) Where is $f(z)$ differentiable and what is its derivative at points where it is differentiable?

   (d) Where is $f(z)$ analytic?

3. (14 pts.) Evaluate the integral $\int_0^{-1+i} z\mathrm{Re}(z^2)\,dz$ along the path C given in the figure at the right.



4. (15 pts.) Evaluate $\oint_C \frac{e^{\pi z}}{z^2 + iz + 2}\,dz$ if C is the following contour taken in a counter-clockwise direction:

   (a) $C : |z| = \frac{3}{2}$ (b) $C : |z - 1| = 1$ (c) $C : |z - i| = 4$.

5. (20 pts.) Evaluate the integral $\int_0^{\pi} \frac{d\Theta}{5 + 4\cos\Theta}$. (Hint: Use the subsitution $z = e^{i\Theta}$.)

6. (5 pts. each)

   (a) Show that the function $u(x, y) = x^2 - (y - 1)^2$ is harmonic in the plane.

   (b) Find the harmonic conjugate of u.

7. (15 pts.) Find a bounded harmonic function $u(x, y)$ in the quadrant $x \leq 0, y \geq 0$ which satisfies the following properties: $u(x, 0) = 5$ for $x < 0$ and $u(0, y) = 2$ for $y > 0$. Justify that your function is harmonic and satisfies the boundary properties. Also evaluate $u(-2, 2)$.