Doctoral Algebra Comprehensive Exam
August 1994

1. Consider the symmetric group $S_4$ on 4 elements.

   (a) Find all Sylow subgroups of $S_4$.

   (b) Give an example of a non-abelian group of order 24 which has a different number of $p$-Sylow subgroups than $S_4$ has, for some $p$.

   (c) Find a composition series for $S_4$.

2. Let $R = k[x, y]$ be the polynomial ring in 2 variables over a field $k$. Consider the ideal $I = (xy - x, y^2 - y)$. Let $S = R/I$ be the quotient ring.

   (a) Show that $m = (x, y)$ is a maximal ideal of $R$.

   (b) Show that $I \subset m$.

   (c) Let $\bar{m} = m/I \subset S$. Show that $\bar{m}$ is a maximal ideal of $S$.

   (d) Show the isomorphism of rings $S_{\bar{m}} \simeq k$, where $S_{\bar{m}}$ denotes the localization of $S$ at $\bar{m}$.

3. Let $R$ be a PID, and let $F$ be a finitely generated free $R$-module. Prove that any submodule of $F$ is also free. (You may not simply quote the theorem that states this fact.)

4. Consider the polynomial $f(x) = x^4 + x^3 + x^2 + 1$ over the finite field $F_2$ with two elements.

   (a) Find the splitting field $K$ of $f(x)$.

   (b) Show directly that this is a Galois extension, and find the Galois group of $K$ over $F_2$.