Ph.D. COMPREHENSIVE EXAM - TOPOLOGY - AUGUST 1994

1. (12 points) Prove (from the definitions) that every connected, locally path connected space is path connected.

2. (12 points) Prove that every separable metric space satisfies the second axiom of countability.

3. (12 points) Let $\{X_\alpha\}_{\alpha \in J}$ be a family of topological spaces. Let $X = \prod_{\alpha \in J} X_\alpha$. For each $\beta \in J$ define $p_\beta : X \to X_\beta$ by $p_\beta((x_\alpha)_{\alpha \in J}) = x_\beta$. Let $f : A \to X$.

   (a) Suppose X has the product topology. Prove that if $p_\beta \circ f$ is continuous for all $\beta \in J$, then f is continuous.

   (b) Is (a) true if X has the box topology? Give a proof or a counterexample.

4. (12 points) Let X be a simply connected normal space; let A be a subspace of X.

   (a) Prove that if A is homeomorphic to $[0,1]$, then A is a retract of X.

   (b) Prove that if A is homeomorphic to S^1, then A is not a retract of X.

5. (12 points) Let $(\widetilde{X}, p)$ be a covering space of X; let $x \in X$. Suppose $\widetilde{X}$ is compact.

   (a) Prove that $p^{-1}(\{x\})$ is finite.

   (b) Suppose $(\widetilde{X}, p)$ is the universal covering space of X. Prove that $\pi_1(X, x)$ is finite. (Don't just quote the theorem which says that the order of $\pi_1(X, x)$ is equal to the number of sheets of the universal covering space $(\widetilde{X}, p)$; prove it in this special case.)

6. (12 points) Let $X = A \cup B$, where A and B are circles and $A \cap B = \{x\}$. Let f and g be loops based at x such that f runs once around A and g runs once around B. (Draw a picture!) Let α and β be the elements of $\pi_1(X, x)$ determined by f and g, respectively.

   (a) Classify up to isomorphism all the (connected) 2-sheeted covering spaces of X. For each isomorphism class draw a representative $\widetilde{X}$ and indicate how p maps $\widetilde{X}$ to X.

   (b) Choose *one* of the covering spaces from (a). Let $\{\tilde{x}_1, \tilde{x}_2\} = p^{-1}(\{x\})$. Compute the action of $\pi_1(X, x)$ on $p^{-1}(\{x\})$, i.e. find $\tilde{x}_1 \cdot \alpha$, $\tilde{x}_1 \cdot \beta$, $\tilde{x}_2 \cdot \alpha$, and $\tilde{x}_2 \cdot \beta$.

7. (12 points) Let M_g denote the connected sum of g tori; let N_g denote the connected sum of g projective planes. Classify each of the following surfaces as an M_g or N_g, specifying g.

   (a) The connected sum of a torus and a projective plane.

   (b) The compact orientable surface S with Euler characteristic $\chi(S) = -4$.

   (c) The compact non-orientable surface S with Euler characteristic $\chi(S) = -3$.

8. (12 points) Let T be a torus and P a projective plane. Give presentations for the fundamental groups of the following spaces. (Don't just say what the group is algebraically; give a *presentation* for it.)

   (a) $T \# P$, where "#" denotes connected sum.

   (b) $T \vee P$, where "$\vee$" denotes the gluing of the two spaces together at a single point.

   (c) $T \times P$

9. (4 points) State the Seifert-Van Kampen Theorem for the case of a space X which is the union of two subspaces U and V. Be sure to state all the hypotheses.