Master's Comprehensive Exam
Advanced Linear Algebra
August 1995

1. (20 pts.) Let the nonzero linear transformation $T : R^2 \to R^2$ be given by $T(\vec{x}) = A\vec{x}$, for some 2×2 matrix A. For each of the conditions (a) and (b) below, give an example of a nonzero matrix A so that T will satisfy the given condition. Give all possible values of $rk(T)$ in each case. Justify your answers.

   (a) $\ker T \subseteq \operatorname{im} T$

   (b) $\operatorname{im} T \subseteq \ker T$

2. (25 pts.) Let A and B be two $n \times n$ real matrices satisfying that $AB = -BA$ (In this case, we say that A and B *anticommute*.)

   (a) If n is odd, show that one of A or B is not invertible.

   (b) Show that this can be false if n is even.

3. (25 pts.) Let V be finite-dimensional real or complex inner product space with inner product $\langle \cdot, \cdot \rangle$. Let $T : V \to V$ be a self-adjoint linear transformation and let u and v be eigenvectors corresponding to different eigenvalues λ and μ. Prove that u and v are orthogonal.

4. (30 pts.) Let A be the matrix

$$A = \begin{bmatrix} -2 & -1 & 2 \\ -5 & -6 & 16 \\ -3 & -3 & 8 \end{bmatrix}.$$

   The characteristic polynomial of A is $h(x) = \det(xI_3 - A) = (x+1)^2(x-2)$.

   (a) Find an invertible matrix P and a matrix J in Jordan canonical form so that $J = P^{-1}AP$.

   (b) Find the minimal polynomial of A. Explain your reasoning.