Masters Comprehensive Exam

Modern Algebra

August 1995

There are 6 questions in all. Be sure to give complete proofs and explanations.

1. (a) If σ is any permutation in the symmetric group S_n and $(x_1, \dots, x_r)$ is an r-cycle in S_n, show that $\sigma(x_1, \dots, x_r)\sigma^{-1}$ is again an r–cycle. (Determine the resulting r–cycle.)

(b) Are any two r–cycles in S_n conjugate? If so, prove it. If not, give a counterexample.

2. Let p and q be primes with $p < q$.

(a) Determine all abelian groups pf order pq^2.

(b) Is a group of order pq^2 always abelian? If so, prove it. If not, give an example of primes $p < q$ and a non-abelain group of order pq^2.

3. Let V be a vector space over a field F. Let P be a linear transformation $P : V \to V$ such that $P^2 = P$, that is, a projection.

(a) Show that $V = \Im P \oplus \ker P$.

(b) If V is finite dimensional and P and Q are two commuting projections, so $PQ = QP$, show that V has a basis with respect to which both P and Q are represented by diagonal matrices.

4. Let R be a principal ideal domain (PID) and M a module over R.

(a) Define what it means for M to be a torsion module.

(b) Define what it means for M to be a torsion free module.

(b) Define what it means for M to be a torsion free module. (c) Suppose that M is finitely generated over R and let M_t be the torsion submodule of M. Show that $M = M_t \oplus N$ where N is a free module. (You may use whatever facts you know about free modules over a PID, but be sure to clearly state them when you use them.)

5. Let F be a field and $f(x) \in F[x]$ a polynomial with coeffecients in F. Show that there exists a finite extension K of F in which $f(x)$ has a root. (Construct the extension.)

6. Let $f(x) = x^4 - 25 \in \mathbb{Q}[x]$.

(a) Determine the splitting field E of $f(x)$ and all intermediate fields $\mathbb{Q} \subset F \subset E$.

(b) Determine the Galois G group of $f(x)$ and all of its subgroups H.

(c) Explicitly exhibit the Galois correspondence between all intermediate fields $\mathbb{Q} \subset F \subset E$ and all subgroups $H \subset G$.