Doctoral Algebra Examination

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Give clear and coherent proofs of all your assertions. If you cannot solve a problem completely, state the definitions of all terms in the problem and record any partial progress you may have made.

1. Use the Sylow theorems to prove that any group of order 15 is cyclic.

2. Let G, H, and K be groups.

   (a) Define what it means for a sequence of group homomorphisms

   $$H \xrightarrow{\alpha} G \xrightarrow{\beta} K$$

   to be *exact.*

   (b) Define the term *short exact sequence.*

   (c) Give an example of a short exact sequence

   $$1 \longrightarrow H \xrightarrow{\alpha} G \xrightarrow{\beta} K \longrightarrow 1$$

   where G is *not* isomorphic to the direct product $H \times K$.

3. Let R be a ring.

   (a) Define what it means for an R-module to be *injective.*

   (b) Prove that any homomorphic image of an injective $\mathbb{Z}$-module is injective.

4. Show that the intersection of all prime ideals in a commutative Noetherian ring is nilpotent (i.e. $\mathfrak{a}$ is nilpotent if $\mathfrak{a}^k = 0$ for some integer $k \geq 1$).

(Next page)

5. Let K_1, K_2 be finite extensions of the field k, all of which are contained in the field L.

   (a) Define the *compositum* K_1K_2.

   (b) Prove that $[K_1K_2 : k] \leq [K_1 : k] \cdot [K_2 : k]$.

   (c) Prove that, if the degrees $[K_1 : k]$ and $[K_2 : k]$ are relatively prime, then equality holds in part (b).

6. Let K be a field with p^n elements where p is a prime and n is a positive integer. Show that every element of K has a unique p-th root in K.

7. Let F be a field and let $L = F(x_1, \ldots, x_n)$ be the field of all rational functions in the indeterminates $x_1, \ldots, x_n$ with coefficients in F. Let the symmetric group S_n act on $f(x) \in L$ by $(\sigma \cdot f)(x_1, \ldots, x_n) = f(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$.

   (a) Show that this identifies S_n as a group of field automorphisms of L.

   (b) Let K be the subfield of L fixed by the action of all elements of S_n (K is the field of symmetric functions). Show that L is the splitting field of an irreducible polynomial of degree n over K (i.e. with coefficients in K).

   (c) Prove that L/K is Galois with group S_n.