PhD/EdD Comprehensive Exam
Complex Analysis
August 1995

NOTATION: Ω will denote an open subset of $\mathbb{C}$, the complex plane. $H(\Omega)=$ {holomorphic functions in Ω}, and $D(z,r)=\{w\in\mathbb{C}:|z-w|<r\}$, the open disc with center z and radius r.

Six *complete* solutions will constitute a pass.

1. Suppose f is entire (that is, $f\in H(\mathbb{C})$) and $|f(z)-f(w)|\le|z-w|$ for all $z,w,\in\mathbb{C}$. Show that f is a polynomial.

2. Suppose $f_n\in H(D(0,1))$ for $n=1,2,\ldots.$ Suppose that $f_n\to f$ uniformly in $D(0,1)$ as $n\to\infty$, and f has exactly one zero in $D(0,1/2)$. Show that there exists an N such that f_n has exactly one zero in $D(0,1/2)$ for all $n>N$.

3. True or False (with brief explanation): There exists a sequence of (holomorphic) polynomials P_n such that $P_n(0)=1$ for all n, although $P_n(z)\to 0$ for all $z\in\mathbb{C}$ with $z\neq 0$.

4. Suppose we are given real numbers a,b,c and d, and we define $T:\mathbb{C}\to\mathbb{C}$ by $T(x+iy)=ax+by+i(cx+dy)(x,y$ real). For what values of a,b,c and d will T be holomorphic? (Explain briefly.)

5. Let $\Omega=D(0,1)\setminus\{0\}$. True or False: Any function $f\in H(\Omega)$ is the derivative of some other function $g\in H(\Omega)$. Explain.

6. Let $\Omega=\{x+iy:x,y\in R,0<y<1+x^2\}$. Suppose that $z,w\in\Omega$. Show that there exists a function $f\in H(\Omega)$ such that f is 1-1 in $\Omega, f(\Omega)=\Omega$ and $f(z)=w$. (In other words, show that Ω has a holomorphic automorphism mapping z to w.)

7. Suppose that Ω is connected and open, $f,g\in H(\Omega)$ are holomorphic in Ω and the product fg vanishes identically in Ω. Show that at least one of f and g must also vanish in all of Ω.

8. (a) Define "simply connected."

   (b) Recall the following definition: A set $\Omega\subset\mathbb{C}$ is *convex* if the line segment joining any two points of Ω lies in Ω. Now suppose that Ω is a convex open subset of $\mathbb{C}$ and show that Ω is simply connected.

9. Suppose that $\{u_n\}$is a sequence of non-negative functions harmonic in $D(0,2)$. Suppose that $u_n(0)\to\infty$ and show that $u_n(1/2)\to\infty$.

   (Note: You're not allowed to use "Harnack's Theorem" (or "Harnack's Inequality") in this one—that would amount to assuming what you are supposed to prove.)