PhD COMPREHENSIVE EXAM—REAL ANALYSIS—August 1995

1. Indicate whether each of the following statements is true or false, and briefly explain your answer.

   a) Every countable subset of $\mathbf{R}$ is Lebesgue measurable, with Lebesgue measure 0.

   b) Every subset of $\mathbf{R}$ of Lebesgue measure zero is countable.

   c) There exists a closed subset of [0,1] with empty interior and Lebesgue measure one.

2. Let $(X, \mathcal{M}, \mu)$ be a measure space, and suppose $\{E_j\}_{j=1}^{\infty} \subset \mathcal{M}$. Show that if $\sum_{j=1}^{\infty} \mu(E_j) < \infty$, then $\mu(F) = 0$, where $F = \cap_{k=1}^{\infty}\{\cup_{j=k}^{\infty} E_j\}$.
   Interpretation: $F = \{x: x \text{ belongs to infinitely } E_j\}$, so $\mu(F) = 0$ says that almost every $x \in X$ belongs to at most finitely many E_j.

3. a) State carefully the Monotone Convergence Theorem, Fatou's Lemma, and the Dominated Convergence Theorem.

   b) Prove one of the three theorems in part a); you may assume either of the other two in your proof.

   c) Evaluate with explanation

$$\lim_{n\to\infty} \int_0^{\infty} \frac{n\sin(x/n)}{x(1+x^2)}\,dx\ .$$

4. Suppose that μ and ν are measures on a measurable space $(X, \mathcal{M})$.

   a) Define what it means for μ to be absolutely continuous with respect to ν and what it means for μ to be singular with respect to ν.

   b) Prove using only the definitions that if μ is both absolutely continuous and singular with respect to ν then μ is the zero measure.

5. If f is a continuous real-valued function on [0,1], the *moments* of f are the numbers $\int_0^1 f(x)x^n\,dx$ for $n = 0, 1, \ldots$ Use the Stone-Weierstrass Theorem to show that two functions are identical if they have the same sequence of moments.

6. Recall: If X is a Banach space, $\{x_n\}$ is a sequence in X, and $x \in X$, then $x_n \to x$ weakly means that $\phi(x_n) \to \phi(x)$ for all ϕ in the dual of X.

   a) Fix p so that $1 < p < \infty$, and suppose that $f_n \to f$ weakly in $\ell^p(\mathbf{N})$. Show that $f_n \to f$ pointwise, that is, for all $m \in \mathbf{N}$, $f_n(m) \to f(m)$ as $n \to \infty$.

   b) Find a sequence $\{f_n\}$ in $\ell^1(\mathbf{N})$ so that $\|f_n\|_1 = 1$ for all n and $f_n \to 0$ pointwise, but $f_n \not\to 0$ weakly in $\ell^1(\mathbf{N})$.