Master's Comprehensive Exam – Advanced Calculus – August 1996

1. (a) Define "$f : [a, b] \to \mathbf{R}$ is Riemann integrable".

   (b) Give an example of a function which is not Riemann integrable and show, using your definition in part (a), that it is not.

2. Provide a counterexample to the converse of each of these theorems.

   (a) If a set S in $\mathbf{R}$ is compact, then S is bounded.

   (b) If a function $f : \mathbf{R} \to \mathbf{R}$ is differentiable, then f is continuous.

   (c) If $f : [0, 1] \to \mathbf{R}$ is continuous then f is bounded.

3. Show that if $f'(x) > 0$ for all $x \in (a, b)$ then f is strictly increasing on (a, b).

4. What conditions on a and b are sufficient to guarantee that the equation

$$x^2y - y^3 + x^3 - a^2b - a^3 + b^3 = 0$$

   has a solution $y = f(x)$ valid near (a, b)?

5. (a) For a function $f : [a, b] \to \mathbf{R}$ define "f is of bounded variation on $[a, b]$" and define the total variation of f on $[a, b]$.

   (b) Suppose that $f : [a, b] \to \mathbf{R}$ is monotonically increasing. Show that f is of bounded variation on $[a, b]$ and determine it total variation.

6. (a) For a function $f : \mathbf{R} \to \mathbf{R}$ define "f is uniformly continuous on $\mathbf{R}$."

   (b) Assume that $f : \mathbf{R} \to \mathbf{R}$ is bounded and assume that f is Riemann integrable on any finite interval. Define, for each $x \in \mathbf{R}$, $F(x) = \displaystyle\int_0^x f(s)\, ds$. Show that F is uniformly continuous on the set $\mathbf{R}$.

7. Suppose that, for each $n \geq 1$, $f_n : [0, 1] \to \mathbf{R}$ is a continuous function and suppose that the sequence $\{f_n\}$ converges uniformly to a function f on $[0, 1]$. Show that

$$\lim_{n\to\infty} \int_0^1 f_n(x)\, dx = \int_0^1 f(x)\, dx$$

8. Suppose that the series $\displaystyle\sum_{n=0}^{\infty} a_n x^n$ has a radius of convergence equal to r, a finite number. What is the radius of convergence of these two series?

   (a) $\displaystyle\sum_{n=0}^{\infty} n a_n x^n$

   (b) $\displaystyle\sum_{n=0}^{\infty} \frac{a_n}{2^n} x^n$

9. (a) For what values of x does the series

$$\sum_{n=1}^{\infty} (1 - x^2)^n$$

   converge?

   (b) Determine those intervals of the form $[a, b]$ where the series converges uniformly.