MASTER'S COMPREHENSIVE EXAM *ADVANCED LINEAR ALGEBRA*

AUGUST 1996

1.a) Find a basis for the nullspace $\mathbf{V} \subset \mathbf{R}^4$ of the matrix (one row!)

$$A = [\,1 \quad 1 \quad 1 \quad 1\,].$$

b) Find an orthogonal basis for the same nullspace $\mathbf{V} \subset \mathbf{R}^4$.

2. Given are two $n \times 1$-column vectors $u,\ v \in \mathbf{R}^n$. Then $A = I + u\,v^T$ is an $n \times n$-matrix, provided that the transpose v^T is regarded as a $1 \times n$-matrix. Show that A is invertible if and only if $v^T u \neq -1$. Can you find an expression for A^{-1} similar to that for A?

3. Consider the eigenvalues of a real 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Show that a sufficient condition for A to have real eigenvalues is that the entries $b,\ c$ have the same signs (the result can be strengthened slightly).

4. Suppose a linear transformation $T : \mathbf{V} \to \mathbf{V}$ of an n-dimensional complex vectorspace $\mathbf{V}$ has n distinct eigenvalues $\lambda_1, \ldots, \lambda_n,\ \lambda_i \neq \lambda_j$ if $i \neq j$. Find an expression, in terms of eigenvectors of T, for a cyclic vector $v \in \mathbf{V}$ for T, *i.e.*, such that $\{v, Tv, T^2v, \ldots, T^{n-1}v\}$ is a basis of $\mathbf{V}$.

5. Given is an $n \times n$ real symmetric matrix A with the special condition that its characteristic polynomial can be factored in the form $p(\lambda) = \det(\lambda I - A) = \lambda^k q(\lambda)$, where $q(\lambda)$ is a polynomial of degree $n - k$ such that $q(0) \neq 0$. Prove that the rank of A is $n - k$. Your proof should **not** depend on the *Principal Axes Theorem.* **HINT:** Consider an orthonormal basis $\mathcal{B}$ of $\mathbf{R}^n$ that extends an orthonormal basis of the nullspace of A. Then compute the characteristic polynomial $p(\lambda)$ from the matrix representation of the linear transformation corresponding to A in the basis $\mathcal{B}$.