PH. D. COMPREHENSIVE EXAM — ALGEBRA — AUGUST, 1996

Notation: $\mathbf{Z}_n$ denotes the cyclic group of integers modulo n. $\mathbf{F}_p$ denotes the finite field with p elements. The symbols $\mathbf{Q}$, $\mathbf{R}$, and $\mathbf{C}$ denote, respectively, the fields of rational numbers, real numbers, and complex numbers. All rings are assumed to have an identity element.
Grading: Four problems done completely (i. e., full credit on four problems) will guarantee a pass. It is probably wise to attempt more than four, in case you receive only partial credit on some of them.

1. (a) Find all subgroups H of order 4 of the group $\mathbf{Z}_8 \times \mathbf{Z}_2$.
   (b) For each subgroup H from part (a), express $(\mathbf{Z}_8 \times \mathbf{Z}_2)/H$ as a direct product of cyclic groups of prime power order.

2. Let G be a group of order $255 = 3 \cdot 5 \cdot 17$.
   (a) Show that G has a unique 17-Sylow subgroup.
   (b) Show that the 17-Sylow subgroup is contained in the center of G.
   (c) Show that G is abelian.

3. Let R be a commutative ring and let
$$0 \to M_1 \to M_2 \to M_3 \to 0$$
   be an exact sequence of R-modules. Prove that M_2 is Noetherian if and only if M_1 and M_3 are Noetherian.

4. Let R be a ring, I a minimal left ideal of R (i. e., the only left ideals that I contains are (0) and itself). Suppose $I^2 \neq (0)$. Show that $I = Re$ for some $e \in I$ with $e^2 = e$. (Hint: First show $I = Ia$ for some $a \in I$. Then there exists $e \in I$ such that $ea = a$. Show that this e is the desired element.)

5. Let K be the splitting field of $x^6 + 3$ over $\mathbf{Q}$.
   (a) Compute the Galois group of K over $\mathbf{Q}$.
   (b) Find all fields intermediate between $\mathbf{Q}$ and K, indicating which are Galois over $\mathbf{Q}$ and which are not.

6. Let F be the splitting field of $(x^2 + 1)(x^3 - x - 1)$ over the field K. Find the Galois group of F over K in each of the following cases:
   (a) $K = \mathbf{Q}$ (b) $K = \mathbf{Q}(\sqrt{-23})$ (c) $K = \mathbf{R}$
   (d) $K = \mathbf{C}$ (e) $K = \mathbf{F}_5$