PhD COMPREHENSIVE EXAM—COMPLEX ANALYSIS—August 1996

Notation: $B = \{z \in \mathbf{C}: |z| < 1\}$, the open unit disc

1. a) Let f be an entire function, and fix $p, q \in \mathbf{C}$ with $p \neq q$. If $r > 0$ is chosen so that $|p| < r$ and $|q| < r$, evaluate

   $$\int_{\gamma_r} \frac{f(z)}{(z-p)(z-q)}\, dz,$$

   where $\gamma_r(t) = re^{it}$ on $[0, 2\pi]$.

   b) Use part a) to give a proof of Liouville's Theorem.

2. Let f and g be analytic functions on B so that $f(z)g(z) = 0$ for all $z \in B$. Show that either $f(z) = 0$ for all $z \in B$ or $g(z) = 0$ for all $z \in B$.

3. Show that the polynomial $2z^5 - 6z^2 + z + 1$ has exactly three zeros (counting multiplicities) in $\{z\colon\ 1 < |z| < 2\}$.

4. Suppose u is harmonic on the punctured unit disc $B \setminus \{0\}$. Does there exist an analytic function f on $B \setminus \{0\}$ so that u is the real part of f there? Give a proof or a counterexample.

5. a) State carefully the Riemann Mapping Theorem.

   b) Let $D = \{x + iy\colon\ x, y \in \mathbf{R} \text{ and } x^2 < y^2 + 1\}$ and f an entire function so that $f(\mathbf{C}) \subset D$. Show that f is constant.

6. Let E be a set with no limit point in $\mathbf{C}$. Show that there exists a meromorphic function f on $\mathbf{C}$ so that f has a pole of order two at each point of E and $f \neq 0$ on $\mathbf{C}$.