Master's Comprehensive Exam
Advanced Calculus
August 1997

1. (a) Suppose that $A \subset \mathbb{R}$ and $f : A \to \mathbb{R}$. Define:
   i. "f is continuous on A"
   ii. "f is uniformly continuous on A"

   (b) Give an example of a continuous function $f : (0,1) \to \mathbb{R}$ which is not uniformly continuous. Explain.

   (c) Fill in the blank to make a true statement (no proof required): "Suppose that $A \subset \mathbb{R}$ and $f : A \to \mathbb{R}$ is continuous. If A is a ________ set then f is uniformly continuous on A."

2. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is continuously differentiable (that is, the derivative exists and is continuous). For $n = 1, 2, ...$, define $g_n(x) = \dfrac{f(x + 1/n) - f(x)}{1/n}$. Show that $g_n(x) \to f'(x)$ uniformly for $0 \le x \le 1$ as $n \to \infty$. (Note that the pointwise convergence is just the definition of the derivative, so you should make certain you prove *uniform* convergence. You may use basic facts about continuity and compact sets.)

3. (a) Suppose that $g(s,t)$ is a differentiable function of two variables and let $f(x) = g(x^2, x)$. Give a formula for the derivative $f'(x)$ in terms of the partial derivatives of g.

   (b) Define $f(x) = \int_0^x \sin(x^2 t^2)\, dt$. Find the derivative $f'(x)$. (The answer will involve some definite integrals you cannot and need not evaluate, just as in the definition of f.)

4. Suppose that $f : \mathbb{R}^2 \to \mathbb{R}$.

   (a) Define what it means to say that f is *differentiable* at the origin.

   (b) Give an example where f is *not* differentiable at the origin, even though both partial derivatives $f_x(0,0)$ and $f_y(0,0)$ exist.

5. Suppose that the series $\sum_{n=1}^{\infty} |a_n|$ converges and show that $\sum_{n=1}^{\infty} a_n$ converges. (In other words, show that absolute convergence implies convergence.)

6. Prove true or false: Any subset of $\mathbb{R}$ is either open or closed.

7. Let $f(x) = \sin(e^{(x^2)} + \cos(1/(1 + x^2)))$. Find $\int_0^1 f'(x)\, dx$.