Master's Comprehensive Exam

General Topology, August 1997

1. (10pts) Let X be the set $\{0, 1, 2, 3, 4\}$ and T be a topology of X given by

$$T = \{X, \emptyset, \{0\}, \{2,3\}, \{0,2,3\}, \{1,2,3,4\}\}.$$

   (i) Find all closed subsets of X.

   (ii) Find interior points and exterior points of $A = \{1, 2, 3\}$.

   (iii) Find the closure of A. Is (X, T) a Hausdorff space ?

2. (20pts) Prove that a completely regular space is regular.

   Recall: A topological space X is completely regular if it satisfies the following. For a closed subset F of X and $p \in X \setminus F$, there exists a continuous function $f : X \to [0, 1]$ such that $f(p) = 0$ and $f(F) = 1$.

3. (20pts) Let $f : (X, T_X) \to (Y, T_Y)$ be a one-to-one continuous function. If (X, T_X) is compact and (Y, T_Y) is Hausdorff, then (X, T_X) is homeomorphic to $(f(X), T_Y|_{f(X)})$ ($T_Y|_{f(X)}$ is the induced topology of $f(X)$).

4. (20pts) If A is a totally bounded subset of a metric space X, then every sequence $\{a_n\}$ in A contains a Cauchy subsequence.

5. (20pts) Let $f : [0, 1] \to [0, 1]$ be a continuous function. Show that there exists a point $a \in [0, 1]$ such that $f(a) = a$.

6. (10pts) Show that every regular Lindelöf space is normal.