Doctoral Algebra Comprehensive Exam
August 1997

1. The goal of this group theory problem is to complete the outline of a proof that all groups G of order 6 are isomorphic to either $\mathbf{Z}_6$ or S_3, proving intermediate results along the way.
   (a) Prove that any group of prime order is cyclic.
   (b) Let G be a group, and A a normal abelian subgroup. Show that G/A operates on A by conjugation.
   (c) Let G be a group of order 6.
      i. Prove that G has a unique 3-Sylow subgroup $H \subset G$.
      ii. Prove that H is a normal subgroup of G, with $G/H \simeq \mathbf{Z}_2$ and $H \simeq \mathbf{Z}_3$.
      iii. Consider the action of $\mathbf{Z}_2$ on $\mathbf{Z}_3$ resulting from combining the results of Problem 1b and Problem 1(c)ii. Show that there are only two possible actions of $\mathbf{Z}_2$ on $\mathbf{Z}_3$, the "trivial action" defined by $\bar{n} \cdot \bar{m} = \bar{m}$ for any $\bar{n} \in \mathbf{Z}_2$ and $\bar{m} \in \mathbf{Z}_3$, and the "nontrivial action" defined by $\bar{n} \cdot \bar{m} = \overline{(-1)^n m}$. (As usual, $\overline{n}$ denotes the equivalence class of $n \in \mathbf{Z}$.)
      iv. Finally, show that if this action is trivial, then $G \simeq \mathbf{Z}_6$, while if this action is nontrivial, then $G \simeq S_3$.

2. Let $D \geq 1$ be an integer, and let $R = \{a + b\sqrt{-D} \mid a, b \in \mathbf{Z}\}$.
   (a) Show that R is a ring.
   (b) Show that if $D \geq 2$, then the only units in R are ± 1.
   (c) Give an example of a $D \geq 1$ such that R is a factorial ring (also called a unique factorization ring). Prove that the resulting R is indeed factorial.
   (d) Give an example of a $D \geq 1$ such that R is *not* a factorial ring. Give an element $a \in R$ and two factorizations into irreducible elements which are distinct (even after allowing reordering and multiplication by units).

3. Let M, M', N be modules over a commutative ring R. Prove that there is an isomorphism of abelian groups

$$\mathrm{Hom}_R(M \oplus M', N) \simeq \mathrm{Hom}_R(M, N) \times \mathrm{Hom}_R(M', N).$$

4. Consider the field extension $\mathbf{Q}(i, 2^{1/4})$ of $\mathbf{Q}$.

   (a) What is the degree $[\mathbf{Q}(i, 2^{1/4}) : \mathbf{Q}]$ of this extension?

   (b) Prove that this is a Galois extension.

   (c) Find its Galois group.

   (d) Illustrate the fundamental theorem of Galois theory by listing all subgroups of the Galois group and all intermediate fields, indicating how these lists are identified by Galois theory.