Comprehensive Exam
Advanced Calculus
August 1998

This exam consists of seven problems to be completed in two hours. A passing score will consist of five problems completed correctly. Please show all work.

1. Let (M, d) be a metric space.
   (a) Define what it means for a subset A of M to be *compact* in M.
   (b) Prove that the union of a finite number of compact subsets of M is compact.
2. (a) Define what it means for a function f to be *uniformly continuous* on an interval $S \subseteq \mathbb{R}$.
   (b) Let $f(x) = \frac{1}{x}$, let $S_1 = [1, 2]$, and let $S_2 = (0, 2]$. Determine, with proof, whether or not f is uniformly continuous on S_1 and S_2.
3. Consider the real-valued double sequence $\{ a_{p,q} \mid p, q \in \mathbb{N} \}$. Recall that such a sequence has a limit L, written $\lim_{p,q\to\infty} a_{p,q} = L$, if for every $\epsilon > 0$, there exists N such that $n, m > N$ imply that $|a_{n,m} - L| < \epsilon$.
   (a) If the double limit $\lim_{p,q\to\infty} a_{p,q}$ exists and equals L, prove that
   $$\lim_{q\to\infty} \left(\limsup_{p\to\infty} a_{p,q} \right) = L.$$
   (b) Give an example of a double sequence $\{a_{p,q}\}$ where the double limit $\lim_{p,q\to\infty} a_{p,q}$ exists but the iterated limit
   $$\lim_{q\to\infty} \left(\lim_{p\to\infty} a_{p,q} \right)$$
   does not exist.
4. Suppose that $f : \mathbb{R} \to \mathbb{R}$ has a finite third derivative f''' in $[a, b]$, and suppose that
   $$f(a) = f(b) = f'(a) = f'(b) = 0.$$
   Prove that $f'''(c) = 0$ for some $c \in (a, b)$.
5. Show that the series
   $$\sum_{n=0}^{\infty} x^n (1 - x)^2$$
   converges uniformly on $[0, 1]$.
6. Prove that
   $$\lim_{n\to\infty} \int_0^{\frac{1}{n}} \cos(nt^2)\, dt = 0.$$
7. Let f be a real-valued function defined on an open neighborhood of $(1, 0)$ in $\mathbf{R}^2$ satisfying $f(1, 0) = 1$. State conditions on f that will ensure the existence of a continuously differentiable function g satisfying $g(1, 1) = 0$ and the condition
   $$f(xy, xg(x, y)) + f(e^{g(x,y)}, yg(x, y)) = 2$$
   in some neighborhood of $(1, 1)$.