Masters Algebra Examination
Oklahoma State University
Department of Mathematics
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1. Let S_n denote the symmetric group on n objects.
   (a) Give exactly one representative of each conjugacy class of S_6.
   (b) Consider the element $\sigma = (1 \quad 2 \quad 3)(4 \quad 5 \quad 6)$ of S_6 (written in its disjoint cycle decomposition). Determine:
      i. the order of σ;
      ii. the order of the centralizer $Z(\sigma)$ of σ in S_6;
      iii. the size of the conjugacy class of σ;
      iv. the structure of the subgroup $Z(\sigma)$ of S_6.

2. Consider groups of order $1998 = 2 \times 3^3 \times 37$.
   (a) Determine all Abelian groups of order 1998.
   (b) Can a group of order 1998 be simple? Prove your answer.
   (c) Can a group of order 1998 be non-Abelian? Prove your answer.

3. Let V be a vector space of dimension n over a field F. Let

$$0 = W_0 \subset W_1 \subset \cdots \subset W_{n-1} \subset W_n = V$$

   be a sequence of subspaces such that $\dim(W_i) = i$ for $0 \leq i \leq n$. Suppose $T : V \to V$ is a linear operator for which each W_i is an invariant subspace. Prove there is a basis $\mathcal{B}$ of V such that the matrix of T with respect to $\mathcal{B}$ is upper triangular.

4. (a) State the definitions of Principal Ideal Domain and Unique Factorization Domain.
   (b) Give examples of each of the following:
      i. An integral domain which is not a unique factorization domain. Justify.
      ii. A unique factorization domain which is not a principal ideal domain. Justify.

5. For each of the following groups G, give a field extension K of $\mathbb{Q}$ that is Galois over $\mathbb{Q}$, with Galois group $G_{K/\mathbb{Q}}$ of K over $\mathbb{Q}$ isomorphic to G. Prove your claims.

$$G = \mathbb{Z}/2\mathbb{Z}\,, \quad \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}\,, \quad S_3\,, \quad D_4$$