Comprehensive Examination
Advanced Calculus
August 1999

A passing mark will be given if 5 problems are completely solved.

1. Let (X, d) be a metric space.
   (a) Define $(x_n)_{n=1}^{\infty}$ *is a Cauchy sequence* in X.
   (b) Give an example of a continuous real-valued function f on $(0,1)$ and a Cauchy sequence $(x_n)_{n=1}^{\infty}$ in $(0,1)$ such that $(f(x_n))_{n=1}^{\infty}$ is not Cauchy.
2. Let $f_n(x) = \dfrac{nx}{1+nx}$, for $n = 1, 2, \ldots$.
   (a) Prove that for any $a > 0$, $(f_n)_{n=1}^{\infty}$ converges uniformly on $[a, \infty)$.
   (b) Prove that $(f_n)_{n=1}^{\infty}$ does not converge uniformly on $[0, \infty)$.
3. Suppose a, b, c, d are real numbers such that $a < b$ and $c < d$. Suppose that $f : (a,b) \times (c,d) \to \mathbb{R}$ is C^1 and $\dfrac{\partial f}{\partial x}(s,t) = 0$ and $\dfrac{\partial f}{\partial y}(s,t) = 0$ for all $s \in (a,b)$, $t \in (c,d)$. Prove that f is constant on $(a,b) \times (c,d)$.
4. Consider the series
   $$\sum_{n=1}^{\infty} \frac{(1+(-1)^n)^n}{n^2}(x-3)^n.$$
   Determine the interval of convergence including convergence at the end points. Justify your answer.
5. Let $(a_n)_{n=1}^{\infty}$ be a sequence of real numbers and let A be the set of all real numbers c such that there is a subsequence $(a_{n_k})_{k=1}^{\infty}$ with $\lim_{k\to\infty} a_{n_k} = c$. Prove or disprove each of the following.
   (a) If $\sup\{a_n : n \in \mathbb{N}\} = s$ is finite, $s \in A$.
   (b) If $\sup\{|a_n| : n \in \mathbb{N}\} < \infty$, then $\liminf_{n\to\infty} a_n = \inf A$.
   (c) If A is a singleton, $\lim_{n\to\infty} a_n$ exists.
6. Let A be a subset of a metric space (M, d). Prove that the closure of A, i.e., the smallest closed set containing A, is the union of the interior of A and the boundary of A.
7. If f is a continuous real valued function on $\mathbb{R}$ such that $f'(x)$ exists for all $x \neq 0$ and $\lim_{x\to 0} f'(x) = 3$, then $f'(0)$ exists.