MASTERS COMPREHENSIVE EXAM
ADVANCED LINEAR ALGEBRA
AUGUST 1999

(1) Let W be the intersection of the three 'hyperplanes'

$$\begin{aligned} 3x_2 + x_3 + 2x_4 - x_5 - x_6 + 4x_7 &= 0, \\ x_1+3x_2 + x_3 + x_4 + 2x_5 - x_6 - 4x_7 &= 0, \\ x_1-6x_2 - 2x_3 - 2x_4 + 5x_5 + 2x_6 - 16x_7 &= 0, \end{aligned}$$

in $\mathbb{R}^7$. What is the dimension of W? Explain your answer.

(2) Let V be a finite dimensional vector space with an inner product (denoted by $(\ ,\ )$). Fix two vectors w_1, w_2 that are orthogonal and of length one. Define a function $T : V \to V$ by

$$T(v) = (v, w_1)w_1 + (v, w_2)w_2.$$

(a) Show that T is a linear transformation.
(b) Find $T(w_1), T(w_2)$ and $T(w)$, for any w orthogonal to both w_1 and w_2.
(c) Describe T geometrically.
(d) Determine the minimal polynomial of T.
(e) Determine the Jordan form of T.
(f) What is the rank of T?

(3) Suppose $T : \mathbb{R}^5 \to \mathbb{R}^5$ is a linear transformation with characteristic polynomial $(x-3)^5$.
(a) Find the determinant of T.
(b) Can you find two linearly independent vectors $v_1, v_2 \in \mathbb{R}^5$ so that $T(v_1)$ and $T(v_2)$ are linearly dependent? Explain your answer.

(4) Let S, T be linear transformations of a vector space V. Suppose that S is invertible and $TS = ST^4$. Show that if v is an eigenvector for T then $S(v)$ is also an eigenvector for T.