MASTERS COMPREHENSIVE EXAM—COMPLEX VARIABLES—August 1999

1. (a) Find all values of $\log\left(2 - 2\sqrt{3}\,i\right)$.

   (b) Find all values of $\left(2 - 2\sqrt{3}\,i\right)^{i}$.

2. (a) Let $f(z) = z^2 + \bar{z}$. Prove that $f(z)$ is nowhere analytic.

   (b) Indicate whether the given function is analytic on the set where it is defined. No proof is required.

   (i) $e^{2z} - z$
   (ii) $|z|^2$
   (iii) $\sin(\bar{z})$
   (iv) $\operatorname{Arg}(z)$ (the principal value of the argument function)
   (v) $\cos\left(z^2 - e^z\right)$

3. Let $u(x, y) = x^3 + x - 3xy^2$.

   (a) Show that u is harmonic on the complex plane.

   (b) Find all the harmonic conjugates of u.

   (c) Find an analytic function f so that $\operatorname{Re} f = u$, and find the Taylor series of f about the point 0.

4. Suppose that f is an entire function with period 1 and period i. (That is, f is analytic in the entire plane and $f(z+1) = f(z+i) = f(z)$ for all z.) Show that f is constant.

5. Use the Residue Theorem to compute with justification the following integral:

$$\int_{-\infty}^{\infty} \frac{\cos x \, dx}{x^2 + 2x + 2}.$$