Ph.D. Algebra Examination
Oklahoma State University
Department of Mathematics
Fall, 1999

Instructions: You must provide complete proofs of all your assertions except where explicitly instructed otherwise. Proofs should be written in clear and grammatical language. All notation and terminology should be clearly defined.

1. Let G be a **simple group** of order $168 = 2^3 \cdot 3 \cdot 7$.

   (a) Citing the Sylow theorems, determine the number of subgroups of G of order 7.

   (b) If H is any subgroup of order 7 of G, prove that the **normalizer** $N(H)$ of H in G has order 21.

   (c) Prove that the normalizer $N(H)$ is isomorphic to the group given by the presentation
   $$\langle x, y \mid x^7 = y^3 = 1, \quad yxy^{-1} = x^2 \rangle$$

2. Let G be a group, and let $\sigma_1, \ldots, \sigma_n$ be distinct homomorphisms of G into the multiplicative group $F^\times$ of nonzero elements of the field F. Suppose there are constants $c_1, \ldots, c_n \in F$ such that
   $$c_1\sigma_1(g) + c_2\sigma_2(g) + \cdots + c_n\sigma_n(g) = 0$$
   for all $g \in G$. Prove that $c_1 = c_2 = \cdots = c_n = 0$.

3. Let M be an Artinian module over a ring R. Prove that if $\phi : M \to M$ is an injective homomorphism, then ϕ is an isomorphism.

4. Let R be a commutative ring with identity, and $S \subset R - \{0\}$ a multiplicative subset. Define the localization R_S. If M is an R-module, define the localization M_S of M. Show that $M_S = R_S \otimes_R M$ as R_S-modules.

5. Let $a \in K$ and $b \in L$ be algebraic elements of two extension fields of the field F. Prove there is an F-isomorphism of fields $\beta : F(a) \to F(b)$ which sends a to b if and only if the irreducible polynomials for a and b over F are equal.

6. Let $\zeta = e^{\pi i/10}$, and $K = \mathbb{Q}(\zeta)$.

   (a) Prove that $K/\mathbb{Q}$ is Galois and find its Galois group G.

   (b) Prove that K contains precisely three quadratic extensions of $\mathbb{Q}$ and determine them explicitly.