PhD COMPREHENSIVE EXAM—COMPLEX ANALYSIS—August 1999

Notation: $B = \{z \in \mathbf{C}: |z| < 1\}$, the open unit disc

1. Suppose that f is a nonconstant entire function and that $|f(z)| \leq 1 + |z|$ for all z. Show that f is one-to-one.

2. Let $\sum_{n=0}^{\infty} a_n x^n$ and $\sum_{n=0}^{\infty} b_n x^n$ be power series which converge for $-1 < x < 1$. Suppose that, when $x = \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$, then
$$\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} b_n x^n.$$
Show that $a_n = b_n$ for all n.

3. Let f be a nonconstant entire function, and define g by $g(z) = f(1/z)$ if $z \neq 0$. Prove that f is a polynomial if and only if g has a pole at 0.

4. Suppose f is analytic on a neighborhood of $\overline{B}$ and $|f(z)| < 1$ when $|z| = 1$. Use Rouché's Theorem to determine how many fixed points f must have in B. That is, determine how many solutions the equation $f(z) = z$ has there.

5. By definition, an *automorphism* of a connected open set Ω is a one-to-one analytic function mapping Ω onto Ω.

   (a) Let ϕ and ψ be automorphisms of B. Suppose there are two distinct points a and b in B such that $\phi(a) = \psi(a)$ and $\phi(b) = \psi(b)$. Show that $\phi = \psi$ on B.

   (b) Let D be a simply connected domain in $\mathbf{C}$ which is not equal to $\mathbf{C}$, and let ϕ and ψ be automorphisms of D. Suppose there are two distinct points a and b in D such that $\phi(a) = \psi(a)$ and $\phi(b) = \psi(b)$. Show that $\phi = \psi$ on D.

6. Let $\mathcal{F}$ denote the set of all f analytic on B such that
$$\iint_B |f(x+iy)|^2 \, dxdy \leq 1.$$
Show that $\mathcal{F}$ is a normal family (that is, $\mathcal{F}$ is uniformly bounded on compact subsets of B).
Hint: Fix $p \in B$. Show first that
$$|f(p)|^2 \leq \frac{1}{2\pi} \int_0^{2\pi} |f(p + re^{it})|^2 \, dt$$
for $0 < r < 1 - |p|$. It follows from this (how?) that
$$|f(p)|^2 \leq \frac{1}{\pi(1-|p|)^2} \iint_{|x+iy-p|<1-|p|} |f(x+iy)|^2 \, dxdy,$$
and the result follows.