Comprehensive Exam
Real Analysis
August 1999

A passing mark will be given if 5 problems are completely solved.

1. Let $f : [0,1] \to \mathbb{R}$ be a bounded Lebegue measurable function. Prove that for every $\epsilon > 0$ there is a simple function g on $[0,1]$ such that $\|f - g\|_\infty < \epsilon$.
2. Suppose that $(X, \mathcal{A}, \mu)$ is a finite measure space and that $(f_n)_{n=1}^\infty$ is a sequence of real-valued measurable functions on X. For each of the following types of convergence of a sequence $(f_n)_{n=1}^\infty$ to a function f
   U: uniform convergence
   A: convergence almost everywhere
   M: convergence in measure
   indicate which of the others are implied (but do not prove) and in each case for which the implication is false give a counterexample.
3. Let $(A_n)_{n=1}^\infty$ be a sequence of disjoint intervals such that $m(A_n) = 2^{-n}$ for each n, where m is Lebesgue measure, and $\cup_{n=1}^\infty A_n = [0,1]$. Define $f : [0,1] \to \mathbb{R}$ by $f(x) = n$ for all $x \in A_n$. If g is a measurable function, consider the expression
$$T_f(g) = \int_0^1 f(x)g(x)\,dx.$$
   For each of the following Banach spaces of (equivalence classes of) functions X determine whether T_f defines a bounded linear functional on X and justify your answer.
   (a) $X = L_1[0,1]$ (b) $X = L_2[0,1]$ (c) $X = L_\infty[0,1]$ (d) $X = C[0,1]$
4. Suppose that μ is a finite measure on some measurable space $(X, \mathcal{A})$, $f, g \in L_1(\mu)$ and define
$$\nu(A) = \int_A f\,d\mu \qquad \text{and} \qquad \gamma(A) = \int_A g\,d\mu \qquad \text{for all } A \in \mathcal{A}.$$
   (a) Suppose $\nu \perp \gamma$. What is the relationship between f and g?
   (b) Suppose $\nu \ll \gamma$. What is the relationship between f and g?
5. Suppose that $f : [0,\infty) \to \mathbb{R}$ is continuous and bounded. Prove that if
$$\int_0^\infty f(x)e^{-nx}\,dx = 0 \qquad \text{for } n = 1, 2, \ldots,$$
   then $f = 0$.
6. Let $K(x,y) \in L_\infty([0,1] \times [0,1])$ and define
$$Tf(x) = \int_0^1 K(x,y)f(y)\,dy.$$
   (a) Prove or disprove: T is a well defined bounded linear operator on $L_2[0,1]$.
   (b) Let $(A_i)_{i=1}^k$ and $(B_i)_{i=1}^k$ be two finite sequences of measurable subsets of $[0,1]$. If $K(x,y) = \sum_{i=1}^k 1_{A_i}(x)1_{B_i}(y)$, for all x, y, prove that the range of T, $T(L_2[0,1])$, is finite dimensional and deduce that $\overline{T(\{f : \|f\|_2 \le 1\})}$ is compact.