Comprehensive Exam
Advanced Calculus
August 2000

A passing score will consist of five problems completed correctly. Please show all work.

1. Suppose that (a_n) is a nondecreasing sequence of positive real numbers and there exists a positive integer k such that
$$\lim_{m\to\infty} \frac{a_{k\cdot m}}{k\cdot m} = 0.$$
Prove that $\lim_{m\to\infty} \frac{a_m}{m} = 0$.

2. Find the interval of convergence including behavior at the endpoints of the series
$$\sum_{n=1}^{\infty} \frac{1}{ne^n + n^2}(x-2)^n.$$

3. Define
$$f(x) = \sum_{n=1}^{\infty} \frac{1}{x^n + x^{-n}}$$
wherever the series converges.
(a) Determine the domain of f.
(b) Determine the set of points where f is continuous.

4. Suppose that $f : \mathbb{R}^2 \to \mathbb{R}$.
(a) Define f is *differentiable* at (x_0, y_0).
(b) If f is defined by
$$f(x,y) = \begin{cases} e^{-1/(x^2+y^2)} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0), \end{cases}$$
Prove that f is differentiable at $(0,0)$.

5. Let
$$u(x,y) = \frac{x+y}{x-y} \qquad \text{and} \qquad v(x,y) = \frac{xy}{(x-y)^2}.$$
(a) Compute the Jacobian of the map $T(x,y) = (u(x,y), v(x,y))$.
(b) Show that there exists a function $F : \mathbb{R}^2 \to \mathbb{R}$ such that $F(u,v) = 0$.

6. Let $S = \{(\frac{1}{n}, \frac{1}{n}) : n = 1, 2, 3, \ldots\}$ as a subset of $\mathbb{R}^2$ with its usual topology. Show that S is not compact by giving an open cover of S with no finite subcover.

7. (a) Define (using sums) what it means for a real-valued function f on an interval $[a,b]$ to be *Riemann integrable*.

(b) Using the definition prove that the function f defined by

$$f(x) = \begin{cases} n & \text{if } x = \frac{n-1}{n} \text{ for some } n \in \mathbb{N}. \\ 0 & \text{otherwise.} \end{cases}$$

is or is not Riemann integrable on $[0, 1]$.