MASTER'S COMPREHENSIVE EXAMS
Advanced Linear Algebra
Three problems completed correctly out of four will constitute a passing score on this exam.

1. Let V be a complex vector space.
(a) Define what it means for $\langle \cdot, \cdot \rangle : V \times V \to \mathbf{C}$ to be an inner product on V.
(b) Prove the polar identity, that

$$4\langle u, v\rangle = \|u+v\|^2 - \|u-v\|^2 + i\|u+iv\|^2 - i\|u-iv\|^2$$

for all u, $v \in V$.

2. Recall that, for an $n \times n$ matrix $A = (a_{ij})$, we define the trace of A by $\mathrm{Tr}(A) = \sum_{j=1}^{n} a_{jj}$.
(a) Prove that Tr is linear.
(b) Prove that, for any two $n \times n$ matrices A and B, $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$.
(c) Prove that there do not exist $n \times n$ matrices A and B such that $AB - BA = I_n$, where I_n is the $n \times n$ identity matrix.

3. Let A be the matrix

$$A = \begin{bmatrix} 7 & -1 & 3 \\ 0 & 4 & 0 \\ -5 & 7 & -1 \end{bmatrix}.$$

(a) Find the characteristic polynomial of A and the eigenvalues of A.
(b) Find the eigenvectors for A. Is A diagonalizable?
(c) Give the Jordan form J of A.
(d) Find an appropriate basis for $\mathbf{R}^3$ with respect to which the matrix for A is in its Jordan form J, and give the change of basis matrix P so that $J = P^{-1}AP$. (**Note:** you are not required to find the matrix P^{-1}.)

4. Let A and B be two $n \times n$ diagonalizable real matrices.
(a) If A and B have the same characteristic polynomial, prove that they are similar.
(b) If A and B have the same minimal polynomial, need they be similar? Either prove this statement or give a counterexample.