Masters Algebra Examination
Oklahoma State University
Department of Mathematics
Summer, 2000

Instructions: You must provide complete proofs of all your assertions except where explicitly instructed otherwise. Proofs should be written in clear and grammatical language. All notation and terminology should be clearly defined.

1. Let $T : V \to W$ be a linear transformation, where V and W are vector spaces, and assume that V is finite-dimensional. Prove
$$\dim V = \dim(\ker T) + \dim(\operatorname{im} T).$$

2. Let $G = GL_n(\mathbb{R})$ operate on the set $S = \mathbb{R}^n$ by left multiplication.
   (a) Describe the decomposition of S into orbits for this operation
   (b) What is the stabilizer in G of $e_1 = [1, 0, \ldots, 0]^T$?

3. Let H be a subgroup of G. Prove that the following are equivalent.
   (a) H is a normal subgroup of G.
   (b) The operation $(aH)(bH) = abH$ is well defined on the set of cosets of H in G.
   (c) There is a group K and a homomorphism $\alpha : G \to K$ so that $H = \ker \alpha$.

4. If G is a finite group, let C denote its center and for any $g \in G$ let $C(g)$ denote the centralizer of g in G. Explain and prove the *class equation*
$$|G| = |C| + \sum (G : C(g))$$
and use it to show that any group of prime power order has a non-trivial center.

5. Consider the $\mathbb{Z}$-submodule of $\mathbb{Z}^3$ consisting of all (x_1, x_2, x_3) satisfying $x_1 - 2x_2 + x_3 = 0$ and $x_1 + 4x_2 + 9x_3 = 0$. Is it free? If so, find a basis.

6. Let F be a field, E_1 and E_2 two finite extensions of F and $E = E_1E_2$ their compositum. Show that $(E : F) \leq (E_1 : F)(E_2 : F)$. When do you have equality?