MASTERS COMPREHENSIVE EXAM—COMPLEX VARIABLES—August 2000

1. Find the real part and the imaginary part of the principal value of $(1+i)^i$.

2. (a) Find the linear fractional transformation $w = T(z)$ which maps the points $z_1 = \infty$, $z_2 = i$, $z_3 = 0$ onto the points $w_1 = 0$, $w_2 = i$, $w_3 = \infty$, respectively.

   (b) Determine the image of the unit circle under T.

3. Let $u(x, y) = x^3 + x - 3xy^2$.

   (a) Show that u is harmonic on the complex plane.

   (b) Find all the harmonic conjugates of u.

   (c) Find an analytic function f so that $\mathrm{Re} f = u$, and find the Taylor series of f about the point 0.

4. Let C be the unit circle with the counterclockwise orientation. Evaluate the following integrals.

   (a) $\int_C \overline{z}\, dz$

   (b) $\int_C e^{3/z}\, dz$

5. Use the Residue Theorem to compute with justification the following integral, for $a > 0$:

$$\int_0^\infty \frac{1}{(x^2 + a^2)^2}\, dx.$$