MASTER'S COMPREHENSIVE EXAMS

Numerical Analysis

August 2000

A computing machine may be used. Copy all required results from screen. Show ten or more decimal figures. Three problems completely solved will be sufficient for a passing grade.

1. For a positive constant $a > 0$ let x_0 be an approximate squareroot,

$$x_0 \approx \sqrt{a},\ x_0 > 0.$$

Suppose that x_0 has been computed in double precision, and the relative error ϵ of the square x_0^2 is known: $\epsilon = \frac{1}{a}|x_0^2 - a|$. If $\epsilon = 10^{-15}$, determine an accurate estimate of the relative error δ of x_0 itself, $\delta = \frac{1}{\sqrt{a}}|x_0 - \sqrt{a}|$.

Hint: This relates to the condition number $K = \dfrac{a\,f'(a)}{f(a)}$ of the square root function $x = f(a) = \sqrt{a},\ f'(a) = \dfrac{dx}{da}$.

2. One popular method for solving an equation $f(x) = 0$ is the Secant Method. A solution x_* of the equation is given, *i.e.*, $x_* \in \mathbf{R}$ and $f(x_*) = 0$. It is also known that $f(x) \in C^2(\mathbf{R})$, and $f'(x) > 0, \forall x \in \mathbf{R}$.

a) State a definition of the Secant Method in the form of $x_{n+1} = T(x_n, x_{n-1})$, with an operation $x = T(a, b)$ that involves function values $f(a)$ and $f(b)$.

b) Show that the operation $x = T(a, b)$ satisfies the equation $T(a, b) = T(b, a)$.

c) Define the errors $e_m = |x_m - x_*|$. Show that there is an equation $e_{n+1} = M_n\, e_n\, e_{n-1}$ and find an expression in terms of derivatives of $f(x)$ for the factor M_n.

(CONT'D NEXT PAGE • • •)

3.a) Calculate the LU-factors of the matrix $A = \begin{bmatrix} 3 & -1 & 1 \\ 2 & -1 & 1 \\ -1 & 1 & 1 \end{bmatrix} = LU$. We require L to be unit-lower triangular and U upper triangular.

b) By means of the LU-factors from **a)** solve the equations $L\mathbf{y} = \mathbf{b}$, $U\mathbf{x} = \mathbf{y}$, and $A\mathbf{x} = \mathbf{b}$ in that order, where the given right-hand side vector B is $\mathbf{b} = \begin{bmatrix} 6 \\ 5 \\ 6 \end{bmatrix}$. You may not apply Gaussian elimination, pivoting, or the inverse of A to obtain $\mathbf{x}$, but only involve the LU-factors, and the vectors $\mathbf{y}$, b. This process is known as triangular solving or backsubstitution. Show all results, L, U, $\mathbf{y}$, $\mathbf{x}$.

4. A certain quadrature formula with error term exists of the form

$$\int_0^h f(x)dx = A_1 f(0) + A_2 f(\frac{h}{2}) + A_3 f(h) + E(f).$$

It is known that given $f(x) \in C^4([0,h])$, the error $E(f)$ can be expressed in terms of the fourth derivative of the integrand,

$$E(f) = C h^5 f^{iv}(\tilde{x}), \quad 0 < \tilde{x} < h.$$

Determine the constant A_1, A_2, A_3 and C, and, by means of a Taylor series expansion of the integral, prove the validity of the formula.