PhD COMPREHENSIVE EXAM—COMPLEX ANALYSIS—August 2000

Notation: $B = \{z \in \mathbf{C}: |z| < 1\}$, the open unit disc

1. Let D be a connected open set in $\mathbf{C}$ and f a function analytic on D.

   a) Assuming that f has an analytic logarithm in D, show that f has analytic mth roots for all m. Explicitly: Assume that there exists h analytic on D so that $f = e^h$, and show that for each $m \in \mathbf{N}$ there exists a function g_m analytic on D so that $f = (g_m)^m$.

   b) Assume now that $f \not\equiv 0$ and f has analytic mth roots for all m. Show that $f(z) \neq 0$ for all $z \in D$. Hint: Look at multiplicities.

2. (a) Prove that if a, b, c and d are real numbers satisfying $ad - bc = 1$, then

$$M(z) = \frac{az + b}{cz + d}$$

   maps the upper half-plane $\mathbf{H} = \{x + iy: y > 0\}$ conformally onto itself.

   (b) Let $f: B \to \mathbf{C}$ be such that the imaginary part of f is non-negative for all z in B. Assume that f is non-constant and is analytic in B. Show that

   (i) the imaginary part of f is strictly positive for all z in B;

   (ii) if $f(0) = i$, then

$$|f(z)| \leq \frac{1 + |z|}{1 - |z|}.$$

   Hint: Use the Schwarz Lemma.

3. (a) Suppose f is an entire function such that f^2 is a polynomial. Does it follow that f is a polynomial?

   (b) Suppose f is analytic on B and f^2 is a polynomial on B. Does it follow that f is a polynomial on B?

   (Of course, you have to explain your answers).

4. Use residue theory to calculate with justification that

$$\int_0^\infty \frac{x \sin x}{x^2 + 4}\, dx = \frac{\pi}{2e^2}.$$

5. (a) Let G denote an open set in the complex plane containing $\overline{B}$. Suppose f is analytic on G and is non-zero on ∂B. Show that, with the counterclockwise orientation for ∂B,

$$\frac{1}{2\pi i} \int_{\partial B} \frac{f'(z)}{f(z)}\, dz = N(f, B),$$

   where $N(f, B)$ counts the number of zeros of f in B with multiplicity.

   (b) Suppose f_n is a sequence of functions analytic on B such that $f_n \to f$ uniformly on B. Suppose f has exactly R zeros in the set $D = \{z: |z| < \frac{1}{2}\}$ and no zeros on ∂D. Show that there exists an $N > 0$ such that, for all $n > N$, the function f_n has exactly R zeros in D.

6. By definition, an *automorphism* of a connected open set G is a one-to-one analytic function mapping G onto G.

   (a) Let f and g be automorphisms of B. Suppose there are two distinct points P and Q in B such that $f(P) = g(P)$ and $f(Q) = g(Q)$. Show that $f = g$ on B.

   (b) Let D be a simply connected domain in $\mathbf{C}$ which is not equal to $\mathbf{C}$, and let f and g be automorphisms of D. Suppose there are two distinct points P and Q in D such that $f(P) = g(P)$ and $f(Q) = g(Q)$. Show that $f = g$ on D.