Masters Comprehensive Exam - Advanced Calculus - May 1991

1. (4 points each) Provide definitions for the following.

   (a) $E \subset \mathbf{R}^n$ is compact.

   (b) $\lim_{x\to\infty} f(x) = L$

   (c) $f_n(x)$ converges to $f(x)$, uniformly for $x \in E$.

   (d) The metric space $\mathcal{M}$ is *complete.*

2. (10 points) Let $F(t) = \int_0^t \frac{\sin(xt)}{x}\,dx$. Find $F'(t)$

3. (12 points) Let $f(x,y) = xy(1-x-y)$. Find all critical points of f, and classify them as local maxima, local minima, or saddle points.

4. (10 points) Find all values of x for which the series $\sum_{n=0}^{\infty} \frac{nx^n}{n^2+1}$ converges.

5. (10 points) Let $\omega = ydx + zdy - dz$. Find $\int_C \omega$, where C is the curve of intersection of the cylinder $x^2+y^2=1$ with the plane $z=x$.

6. (10 points) Consider the equations

$$\begin{aligned} x_1^2x_2 + x_2^2x_3 + x_3^2x_1 &= 3 \\ x_1^2 + x_2^2 + x_3^2 &= 2x_2 + 2x_3 - 1 \end{aligned}$$

   For which of the variables x_i can these equations be solved for the other two variables x_j and x_k in terms of x_i in a neighborhood of the solution (1,1,1)? It is not necessary to solve the equations.

7. (10 points) Let $T : \mathbf{R}^3 \to \mathbf{R}^3$ be the transformation defined by $T(x,y,z) = (x^2y, y+z, z^2-x^2)$.

   (a) Find the derivative of T.

   (b) Near which points (x,y,z) does T have a local inverse?

8. (6 points each) Which of the following series converge conditionally, which converge absolutely, and which diverge? Explain your reasoning.

   (a) $\sum_{n=1}^{\infty}(-1)^n \frac{2n}{n^2+1}$

   (b) $\sum_{n=1}^{\infty}(-1)^n \frac{(n!)^3}{(3n)!}$

9. (10 points) Let $g(x, y, z) = f(y\sin(x), yz)$. Find g_{12} in terms of the partial derivatives of f.