Ph.D. COMPREHENSIVE EXAM
REAL ANALYSIS
May 1991

Justify your answers. Six complete solutions will constitute a passing score.

1. Suppose that $(\Omega, \mathcal{B})$ is a measurable space and that μ is a finitely additive non-negative real-valued function on $\mathcal{B}$. If $\mu(\Omega) < \infty$, prove that μ is countably additive if and only if for all $(A_n) \subset \mathcal{B}$, $A_n \downarrow \emptyset$, $\mu(A_n) \to 0$. If $\mu(\Omega) = \infty$, is this still true? ($A_n \downarrow \emptyset$ means that $A_n \supset A_{n+1}$ for all n and $\cap_n A_n = \emptyset$.)

2. Consider the following two statements about a Lebesgue measurable function f on $[0,1]$.
   a. f is continuous almost everywhere.
   b. There is a continuous function g such that $f = g$ almost everywhere.
   What is the relationship between a. and b.? (In other words prove or disprove each of the statements a. $\Rightarrow$ b. and b. $\Rightarrow$ a.)

3. Suppose that f and g are functions from $\mathbb{R}$ to $\mathbb{R}$ such that f is Borel measurable and g is continuous. Prove or disprove: $f \circ g$ is Borel measurable.

4. Suppose that p, q, and $r \in [1, \infty)$ and $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$. Show that $fg \in \mathrm{L}_r$ for all $g \in \mathrm{L}_q$ if and only if $f \in \mathrm{L}_p$. You may assume that the measure space is $[0,1]$ with Lebesgue measure.

5. Is the cardinality of the set of Borel subsets of $[0,1]$ the same as the cardinality of the set of Lebesgue measurable subsets of $[0,1]$?

6. Suppose that g is a continuous periodic odd function on $\mathbb{R}$ and that $f \in \mathrm{L}_1([0,1])$. Compute $\lim_{t\to\infty} \int_{\mathbb{R}} f(x) g(tx)\, d\lambda(x)$ where λ denotes Lebesgue measure.

7. Let $P = \{h(x,y) : h(x,y) = f(x)g(y) \text{ for some } f, g \in \mathrm{C}[0,1]\}$. Is the "sup" norm closure of P equal to $C([0,1] \times [0,1])$?

8. Consider the following two conditions on a subset $\mathcal{F}$ of $\mathrm{L}_1^{\mathrm{loc}}(\mathbb{R})$. (Below λ denotes Lebesgue measure.)
   a. For every $\epsilon > 0$ there is a $\delta > 0$ such that for all $f \in \mathcal{F}$, if A is a measurable set with $\lambda(A) < \delta$ then $\int_A |f| d\lambda < \epsilon$.
   b. For every $\epsilon > 0$ there is an $M \in \mathbb{R}$ such that for all $f \in \mathcal{F}$, $\int_{\{x:|f(x)|\geq M\}} |f| d\lambda < \epsilon$.

   Prove or disprove: a. $\Longleftrightarrow$ b.

9.
   a. Carefully state the Radon-Nikodym Theorem.
   b. Suppose $(\Omega, \mathcal{B}, \mu)$ is a measure space with $\mu(\Omega) = 1$ and that $\mathcal{A}$ is a sub-σ-algebra of $\mathcal{B}$. Show that for every $g \in \mathrm{L}_1(\Omega, \mathcal{B}, \mu)$ there is an $\mathcal{A}$-measurable function f on Ω such that for every $A \in \mathcal{A}$, $\int_A f d\mu = \int_A g d\mu$.