Masters Comprehensive Exam

Advanced Calculus

June 1992

Justify all work. Four correct problems (not parts) will constitute a pass.

1.a. Define the terms countable and uncountable set.

b. Let $Q([0,1])$ denote the set of functions on [0,1] with range contained in the rational numbers. Prove or disprove: $Q([0,1])$ is uncountable.

c. Prove or disprove: The set of continuous functions in $Q([0,1])$ is uncountable.

d. Prove that there is a **sequence** of continuous functions $(f_n)_{n=1}^{\infty}$ on [0,1] such that for every continuous function g on [0,1] there is a subsequence of $(f_n)_{n=1}^{\infty}$ which converges to g uniformly. (You may use a certain famous theorem due to Weierstrass.)

2.a. State Taylor's Theorem with (any correct form of the) remainder.

Let $f(x) = \begin{cases} \exp(-\frac{1}{x^2}) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$.

b. Show that $f'(0) = f''(0) = 0$.

Actually $f^{(n)}(0) = 0$ for all n. You may use this fact in the next two parts.

c. Determine the set of all real numbers x_0 such that f has a power series representation (with positive radius of convergence) of the form $\sum_{n=0}^{\infty} a_n(x - x_0)^n$.

d. Show that there is a non-zero C^{∞}-function g on $\mathbb{R}$ such that $g(x) = 0$ for all $x \leq 0$.

3. Suppose that D is the unit square in $\mathbb{R}^2$, i.e., $D = \{(x, y) : 0 \leq x \leq 1 \text{ and } 0 \leq y \leq 1\}$.

a. State Fubini's Theorem for the case of continuous functions $f(x, y)$ on D.

b. Let $D' = \{(x, y, 0) : 0 \leq x \leq 1 \text{ and } 0 \leq y \leq 1\}$. With D identified with D' in the obvious way and $f(x, y)$ as above, consider the surface integral of f over D'. In this context what does the equation $dy\,dx = -dx\,dy$ mean?

c. Explain the apparent contradiction between Fubini's Theorem and the equation in Part b.

4. Consider the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}} \sin(1 + \frac{x}{n}).$$

a. Show that the series converges uniformly on any finite subinterval of $\mathbb{R}^+$.

b. Does the series converge uniformly on $\mathbb{R}^+$?

5. Suppose that (f_n) is a sequence of continuous functions on [0,1] such that $f_n(x) \leq f_{n+1}(x)$ for all $x \in [0, 1]$ and $n \in \mathbb{N}$ and $\lim_{n\to\infty} f_n(x) = g(x) < +\infty$ for all $x \in [0, 1]$.

a. Show that there is an $x_0 \in [0, 1]$ such that $g(x_0) = \inf\{g(x) : x \in [0, 1]\}$.

b. Show that the corresponding result for $\sup\{g(x) : x \in [0, 1]\}$ fails.