Algebra Examination (MS and EdD)

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1. From basic definitions, prove that a subgroup H of a group G is normal if and only if it is the kernel of a group homomorphism $f : G \to J$ for some group J.

2. (a) State the fundamental theorem on the structure of finite abelian groups.

   (b) Determine up to isomorphism all abelian groups of order 32.

3. Let S_n be the symmetric group of degree n acting on the symbols $\{1, 2, \cdots, n\}$.

   (a) Determine the number of 5-Sylow subgroups of S_5.

   (b) Prove that the normalizer of a 5-Sylow subgroup of S_5 has order 20.

   (c) Prove that every subgroup of order 20 in S_5 is the normalizer of some Sylow 5-subgroup of S_5, and deduce how many subgroups of order 20 there are in S_5.

4. Let V be a vector space over the field K, and let T be a linear transformation of V into itself. Prove that any set $\{x_1, \ldots, x_n\}$ of eigenvectors of T corresponding to *distinct* eigenvalues is linearly independent.

5. Let $K[x_1, \ldots, x_n]$ be the ring of polynomials in n variables over the field K.

   (a) Prove that for $n = 1$ the ring $K[x_1]$ is a principal ideal domain.

   (b) Is $K[x_1, x_2]$ a principal ideal domain? Prove your answer.

6. Let R be a principal ideal domain. Let $I_1 \subsetneqq I_2 \subsetneqq I_3 \subsetneqq \cdots$ be a strictly increasing sequence of ideals of R. Prove that this sequence is finite.

7. Find the minimal polynomial of $\sqrt{2} + \sqrt{3}$ over $\mathbb{Q}$ and determine its Galois group.