MASTER'S EXAM, ALGEBRA, JUNE, 1993

NOTATION: If $f : X \to Y$ is a map of sets, $A \subseteq X$, $B \subseteq Y$, then $f(A) = \{f(a) \mid a \in A\}$ and $f^{-1}(B) = \{x \in X \mid f(x) \in B\}$. For n a positive integer, $\mathbf{Z}_n$ denotes the group (or ring) of integers modulo n.

1. Construct a nonabelian group of order 55 (Hint: Use the semi-direct product).

2. Find the elementary divisors and invariant factors of $\mathbf{Z}_4 \oplus \mathbf{Z}_{15} \oplus \mathbf{Z}_{36} \oplus \mathbf{Z}_{100}$.

3. Let G be a group of order $175 = 5^2 \cdot 7$. Show that the 7-Sylow subgroup is normal and lies in the center of G.

4. Let $f : V \to W$ be a linear transformation of vector spaces and let K be the kernel of f.
   (a) If U is a subspace of V, show that $f^{-1}(f(U)) = U + K$.
   (b) If X is a subspace of W, show that $f(f^{-1}(X)) = X \cap f(V)$.

5. Find all maximal ideals and all nilpotent elements of the ring $\mathbf{Z}_{36}$. (An element x of a ring R is called <u>nilpotent</u> if $x^n = 0$ for some positive integer n.

6. (a) Find the Jordan canonical form of the matrix

$$\begin{pmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & -6 \\ 0 & 0 & 1 & -4 \end{pmatrix}.$$

   (b) Find the rational canonical form of the matrix

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

7. (a) Show that $[\mathbf{Q}(\sqrt{2}, \sqrt{3}) : \mathbf{Q}] = 4$ and find a basis for this extension.
   (b) Show that $[\mathbf{Q}(\sqrt{2} + \sqrt{3}) : \mathbf{Q}] = 4$ and find the minimum polynomial of $\sqrt{2} + \sqrt{3}$ over $\mathbf{Q}$.

8. Find the Galois group of the polynomial $x^4 - 4$ over $\mathbf{Q}$.