PhD COMPREHENSIVE EXAM – ALGEBRA – JUNE 1993

1. Determine all groups of order 117 up to isomorphism.

2. Let R be a commutative ring with identity. For any ideal $\mathfrak{a}$ of R, let

$$V(\mathfrak{a}) = \{\mathfrak{p} \mid \mathfrak{p} \text{ is a prime ideal of } R \text{ with } \mathfrak{a} \subset \mathfrak{p}\}$$

$$rad(\mathfrak{a}) = \{x \in R \mid x^n \in \mathfrak{a} \text{ for some integer } n \geq 1\}$$

Show that for any two ideals $\mathfrak{a}$ and $\mathfrak{b}$ of R, $V(\mathfrak{a}) \subset V(\mathfrak{b})$ iff $rad(\mathfrak{a}) \supset rad(\mathfrak{b})$.
(Hint: There is an equivalent description of $rad(\mathfrak{a})$. Prove it and use it.)

3. Prove that an abelian group T is divisible iff it is injective as a $\mathbb{Z}$-module.

4. Prove or disprove each of the following statements. Proofs should be from the definitions.

(a) The polynomial ring $\mathbb{Z}[X_1, \ldots, X_n]$ in a finite number of variables is a Noetherian ring.

(b) The polynomial ring $\mathbb{Z}[X_1, X_2, \ldots]$ in an infinite number of variables is a Noetherian ring.

5. Let F be a finite field.

(a) Prove that its multiplicative group $F^\times$ is cyclic.

(b) Prove that F is perfect.

6. Compute the Galois group of the polynomial $f(x) = x^3 - 2x + 2$ over each of the following fields: (a) $F = \mathbb{Q}$, (b) $F = \mathbb{R}$, (c) $F = \mathbb{F}_5$. In each case describe the lattice of subfields between the ground field F and the splitting field E of $f(x)$.