Ph.D. Comprehensive Exam
Complex Analysis
June 1993

Solve the following problems and justify your answers. Below $D = \{z : |z| < 1\}$. Four essentially complete solutions will constitute a pass.

1. Let $P(x, y)$ be a polynomial with complex coefficients in the real variables x and y. Prove that there exists a polynomial $Q(z)$ such that $P(x, y) = Q(x + iy)$ for all $x, y \in \mathbb{R}$, if and only if $P_y = iP_x$ on $\mathbb{R}^2$. (Here P_y and P_x denote the partials of P with respect to x and y, respectively.)
2. Suppose that $\{z_n\}_{n=0}^{\infty}$ is a sequence of distinct points in D such that $\lim z_n = z_0 = 0$. Let F be analytic in $D \setminus \{z_n\}_{n=0}^{\infty}$.
   a. If there exists an integer N such that $z^N f(z)$ can be defined on $\{z_n\}_{n=0}^{\infty}$ so as to be analytic in a neighborhood of z_0, show that the points z_n are removable singularities or poles of order at most N.
   b. Prove or disprove: If for every $k \in \mathbb{N}$, $z^k f(z)$ cannot be defined on $\{z_n\}_{n=0}^{\infty}$ so as to be analytic in a neighborhood of z_0, then for every $N \in \mathbb{N}$ there exists n such that z_n is a pole of order at least N or an essential singularity.
3. Suppose that f is entire and for all $z \in \mathbb{C}$, if Re $z = 0$ then Re $f(z) = 0$ and if Im $z = 0$ then Im $f(z) = 0$. Prove that $f(z) = -f(-z)$ for all z.
4. Determine the number of zeroes of $z^6 - 5z^4 + 3z^2 - 1$ in $\{z : |z| \leq 1\}$.
5. Let $C_{\rho,\theta}$ be the positively oriented contour $\{z_0 + \rho e^{it} : 0 \leq t \leq \theta\}$.
   a. If f is analytic in the punctured disk $\{z : R > |z - z_0| > 0\}$ and f has a simple pole at z_0, then
$$\lim_{\rho \to 0^+} \int_{C_{\rho,\theta}} f(z)dz = \theta i \mathrm{Res}\{f; z_0\}.$$
   b. Find the principal value of
$$\int_{-\infty}^{\infty} \frac{x}{1 - x^3} dx.$$
6. Suppose that $f(z)$ is a non-constant analytic function in a disk about z_0 and z_0 is a zero of order 2. Prove or disprove: There exists a neighborhood $\mathcal{N}$ of z_0 and an analytic function g such that $g^2 = f$ in $\mathcal{N}$.
7. Let $h(z)$ be a (complex valued) continuous function on the boundary of D and let
$$g(z) = \int_{|w|=1} \frac{h(w)}{w - z} dw$$
   for $z \in D$.
   a. Prove that $g(z)$ is analytic in D.
   b. Compute $g(z)$ if $h(z) = \overline{z}^n$ for some $n \in \mathbb{N}$.
   c. Compute $g(z)$ if $h(z) = f(\overline{z})$ for some f analytic on $\overline{D}$.