PhD COMPREHENSIVE EXAM—REAL ANALYSIS—June 1993

1. Indicate whether each of the following statements is true or false, and briefly explain your answer. In a) and b), E denotes a Lebesgue measurable subset of $\mathbf{R}$, and m denotes Lebesgue measure on $\mathbf{R}$.

   a) If E contains a nonempty open set then $m(E) > 0$.

   b) If $m(E) > 0$ then E contains a nonempty open set.

   c) There exists a continuous increasing function on $[0,1]$ which is differentiable at no point.

   d) Every continuous increasing function on $[0,1]$ is absolutely continuous.

2. Let m denote Lebesgue measure on $\mathbf{R}$, and assume $f \in L^1(m)$ with $f \geq 0$. Show that

   $$(*) \qquad \int f(x+t)\,dm(x) = \int f(x)\,dm(x)$$

   if $t \in \mathbf{R}$. You may assume that $m(E+t) = m(E)$ if E is a measurable set, but you may *not* assume the major theorem of which $(*)$ is a special case.

3. Suppose that μ is a measure, $f \in L^1(\mu)$, and $f \geq 0$. Define $E_n = \{x\colon f(x) \geq n\}$.

   a) Show that $\lim_{n\to\infty} \int_{E_n} f\,d\mu = 0$.

   b) Use a) to show that $\lim_{n\to\infty} n\mu(E_n) = 0$.

4. Let X be a nonempty set and μ^* an outer measure on X.

   a) If $E \subset X$ and $\mu^*(E) = 0$, show that E is μ^*-measurable. (Recall that E is said to be μ^*-measurable if

   $$\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \setminus E)$$

   for all $A \subset X$.)

   b) Let $\mathcal{M}$ denote the collection of μ^*-measurable sets and μ the restriction of μ^* to $\mathcal{M}$. Assuming that μ is a measure, show that it is complete.

5. a) Suppose f is a real-valued continuous function on $[0,1]$ and $\epsilon > 0$. Show that there exist real numbers $a_0, a_1, \ldots, a_n$ so that, for all $x \in [0,1]$,

   $$|a_0 + a_1x^2 + a_2x^4 + \cdots + a_nx^{2n} - f(x)| < \epsilon.$$

   b) Is the result of a) true if $[0,1]$ is replaced by $[-1,1]$? Explain your answer.

6. Recall: If X is a Banach space, $\{x_n\}$ is a sequence in X, and $x \in X$, then $x_n \to x$ weakly means that $\phi(x_n) \to \phi(x)$ for all ϕ in the dual of X.

   a) Fix p so that $1 < p < \infty$, and suppose that $f_n \to f$ weakly in $\ell^p(\mathbf{N})$. Show that $f_n \to f$ pointwise, that is, for all $m \in \mathbf{N}$, $f_n(m) \to f(m)$ as $n \to \infty$.

   b) Find a sequence $\{f_n\}$ in $\ell^1(\mathbf{N})$ so that $\|f_n\|_1 = 1$ for all n and $f_n \to 0$ pointwise, but $f_n \not\to 0$ weakly in $\ell^1(\mathbf{N})$.