Advanced Calculus
Masters Exam
May 30, 1995

1. Below R and R^2 have the topology induced by the usual Euclidean metric $d(\cdot,\cdot)$.

   (a) Suppose that G is an open set in R. Prove that G is a countable union of disjoint open intervals.

   (b) Suppose that G is an open set in R^2. Prove that G is a countable union of open rectangles. (An open rectangle is a set of the form $(a,b)\times(c,d)$.)

   (c) If A_1 and A_2 are subsets of R such that $\inf\{d(x,y) : x \in A_1 \text{and } y \in A_2\} = 0$, prove or disprove: $A_1 \cap A_2 \neq \emptyset$.

2. Let $f : S \to R^m$ where S is an open subset of R^n.

   (a) An informal way of expressing the notion of differentiability of f at a point x_0 in S is to say that there is a unique affine (translate of a linear) function which is the best approximate of f near x_0. Give a precise definition of the differentiability of f at x_0 and explain the informal definition in terms of the precise definition.

   (b) State the chain rule for functions of several real variables.

   (c) Use the chain rule to compute the differential of $f\circ g$ at $(1,-1)$, where $f(x,y,z) = (x+y, xyz)$ and $g(x,y) = (x-y, x+y, xy)$.

   (d) Does $f\circ g$ have an inverse near $(1,-1)$? Why or why not?

3. Prove: If (a_n) is a non-increasing sequence of non-negative real numbers such that $\sum_{n=1}^{\infty} 2^n a_{2^n}$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

4. Suppose that (f_n) is a sequence of continuous real valued functions on $[0,1]$ such that $f_n(t) \leq f_{n+1}(t) \leq 1$ for all $t \in [0,1]$, $n \in N$.

   (a) Prove that (f_n) converges pointwise to a function f.

   (b) Prove that for each $t \in [0,1]$, $\liminf_{s\to t} f(s) \geq f(t)$.

   (c) Prove or disprove: f is continuous.

5. Suppose that $f \in C^2[0,1]$ and that $f''(t) \geq 0$ for all $t \in [0,1]$. Prove that for all $s,t \in [0,1]$, $f(s)+f(t) \geq 2f(\frac{s+t}{2})$.

6. Define the line integral of an R^2-valued function defined on a curve in R^2 (Specify the class of curves as well as the integral itself.) and use the definition to compute the integral of $f(x,y) = (y,x)$ on the curve consisting of the line segment from $(0,0)$ to $(1,2)$.