Master's Comprehensive Exam
Advanced Linear Algebra
May 1995

1. Suppose $T : V \to W$ is a linear map between finite dimensional vector spaces over any field.

   (a) What is the matrix A_T of T with respect to bases of V and W (give definition).

   (b) Prove that T is invertible in A_T is invertible.

   (c) Prove the converse of (b).

   (d) If $V = W = \{$ real polynomials of degree $\leq n\}$ and $T = \frac{d}{dx}$ then five the matrix of T with respect to your favorite basis.

2. For (a) and (b) consider the 2×2 matrix

$$B = \begin{pmatrix} 1 & 5 \\ 0 & -1 \end{pmatrix}.$$

   (a) Find the Jordan form B.

   (b) Find B^{100}.

   (c) How many similarity (i.e. conjugacy) classes of nilpotent 5×5 matrices are there?

3. Suppose $p(x)$ is a polynomial with complex coefficients and A is an $n \times n$ matrix.

   (a) Prove that if λ is an eigenvalue of A then $p(\lambda)$ is an eigenvalue of $P(A)$.

   (b) For which of the following polynomials is $p_k(C)$ invertible for

$$C = \begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 3 \\ 0 & 0 & 1 \end{pmatrix}?$$

   $p_1(x) = x^8 + x^6 + x^4 + x^2 + 1, p_2(x) = x^5 + x^4 + x^3 + x^2 + x + 1, p_3(x) = x^{20} + x^{10} + 1$. Explain your answer.

4. Suppose $S, T : V \to V$ are linear transformations. Prove: If $ST = TS$ then the eigenspaces for S are invariant under T (i.e. If V_λ is an eigenspace for S, then $T(V_\lambda) \subseteq V_\lambda$).