PhD/EdD Comprehensive Exam
Complex Analysis
May 1995

NOTATION: Ω always denotes a connected open subset of $\mathbb{C}$ (perhaps not simply connected) and $H(\Omega)$ is the class of all functions holomorphic (or "analytic") in Ω.

1. (a) Define "isolated singularity."
   (b) Define "removable singularity," "pole," "essential singularity."
2. Suppose that f is an entire function (that is, f is holomorphic in $\mathbb{C}$) and
$$\int_0^{2\pi} |f(re^{it})|dt \leq 1$$
for all $r > 0$. Show that f is constant.
3. Give an example of a non-vanishing holomorphic function with no logarithm. (In other words, give an example of a connected open set Ω and a function $f \in H(\Omega)$ such that f has no zero in Ω but there does not exist $g \in H(\Omega)$ with $f = e^g$). Explain briefly.
4. Suppose $f \in H(\Omega)$. Under what conditions does it follow that there exists $F \in H(\Omega)$ with $f = F'$? (We know that this happens for *all* $f \in H(\Omega)$ if Ω is simply connected. Now what if Ω is not simply connected? Whether Ω is simply connected or not, we know that there exists $F \in H(\Omega)$ with $f = F'$ if and only if f and Ω satisfy exactly *what* condition? Explain briefly.)
5. (a) State Cauchy's theorem for simply-connected domains.
   (b) State the general form of Cauchy's theorem.
6. (a) Define "normal family."
   (b) Suppose that $f_n \in H(\Omega)$ for $n = 1, 2, ..., |f_n(z)| \leq 1$ for all $z \in \Omega$ and the sequence $(f_n(z))$ converges to $f(z)$ for all $z \in \Omega$. Show that $f \in H(\Omega)$. (NOTE: The result is false without the hypothesis $|f_n(z)| \leq 1$.)
7. Let S be the open unit square, $S = \{x+iy : 0 < x < 1, 0 < y < 1\}$. Let D be the unit disc and suppose that $f : S \to D$ is holomorphic and bijective (the Riemann mapping theorem shows that such a function exists.) Explain briefly why f cannot have an analytic continuation to a neighborhood of the closure of S.